A Supply and Demand Framework for Two Sided Matching Markets

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Large Matching Markets

- A large matching market – many students, each college has many seats
- A simpler matching model
  - Supply and demand characterization of stable matching
  - Continuum of students (Aumann 1964)
- Allows for simple derivation of stable outcomes, or derivation of comparative statics (like Becker 1973) while allowing for complex preferences and no transfers (like Gale-Shapely 1962)

The Standard Matching Model

- Colleges $c \in \{1, \dots C\}$
- Students $s \in \{s_1, \dots, s_N\}$
- Agents have strict (responsive) preferences over the other side and being unmatched

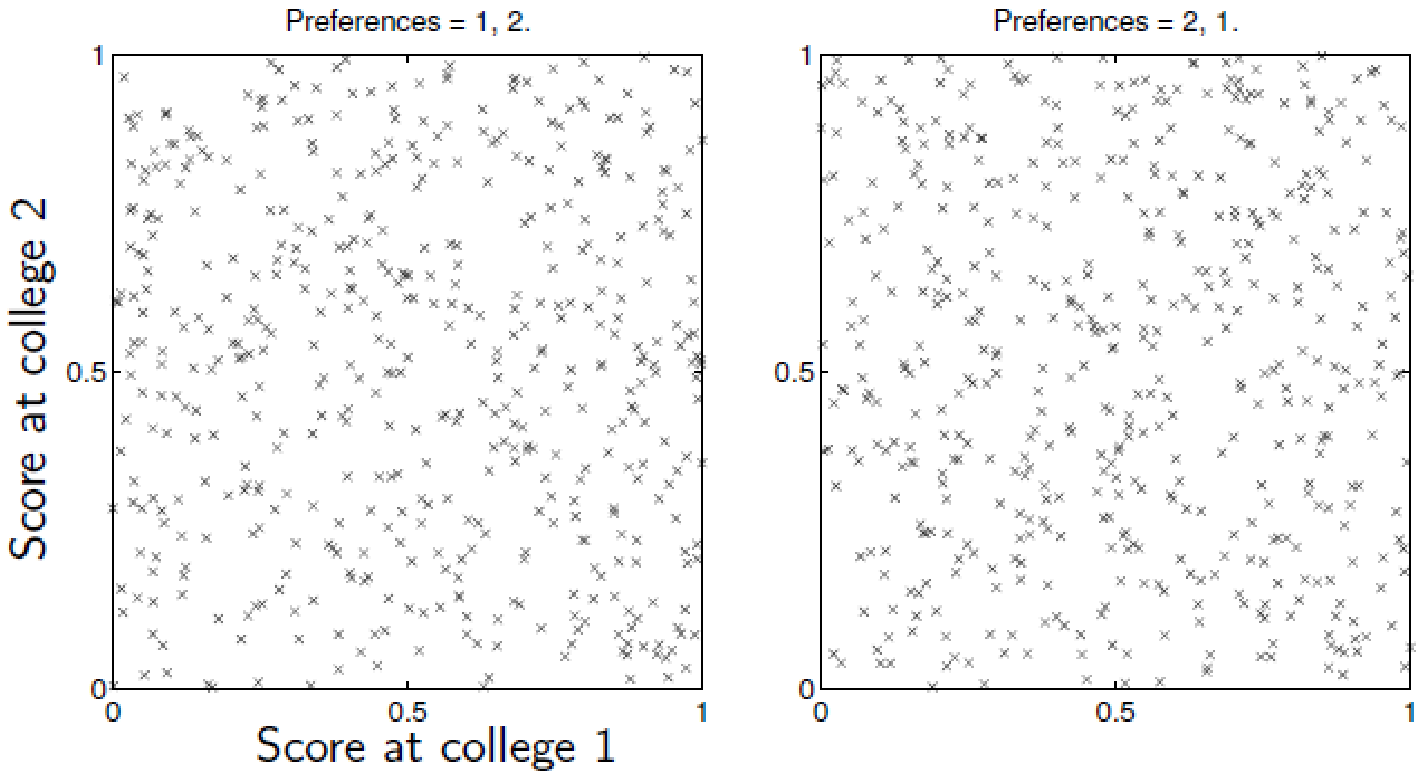
Colleges

1	2
S1	S5
S3	S4
S4	S2
S5	S1
S2	S3

Students

Student	$\succ$
S1	1,2
S2	2,1
S3	1,2
S4	1,2
S5	2,1

When there are many students



$$N = 1,000 \quad S_1 = 250 \quad S_2 = 500$$

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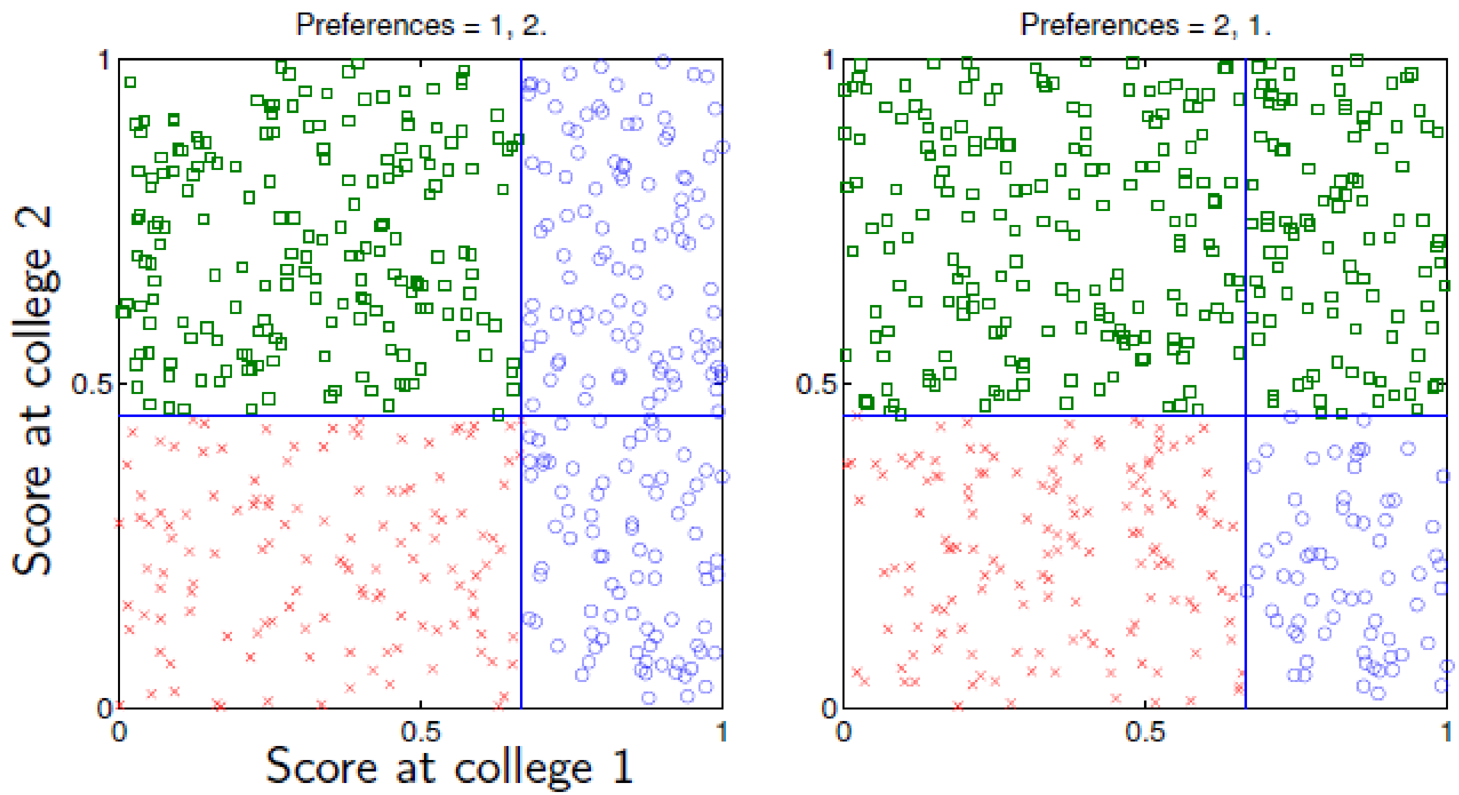

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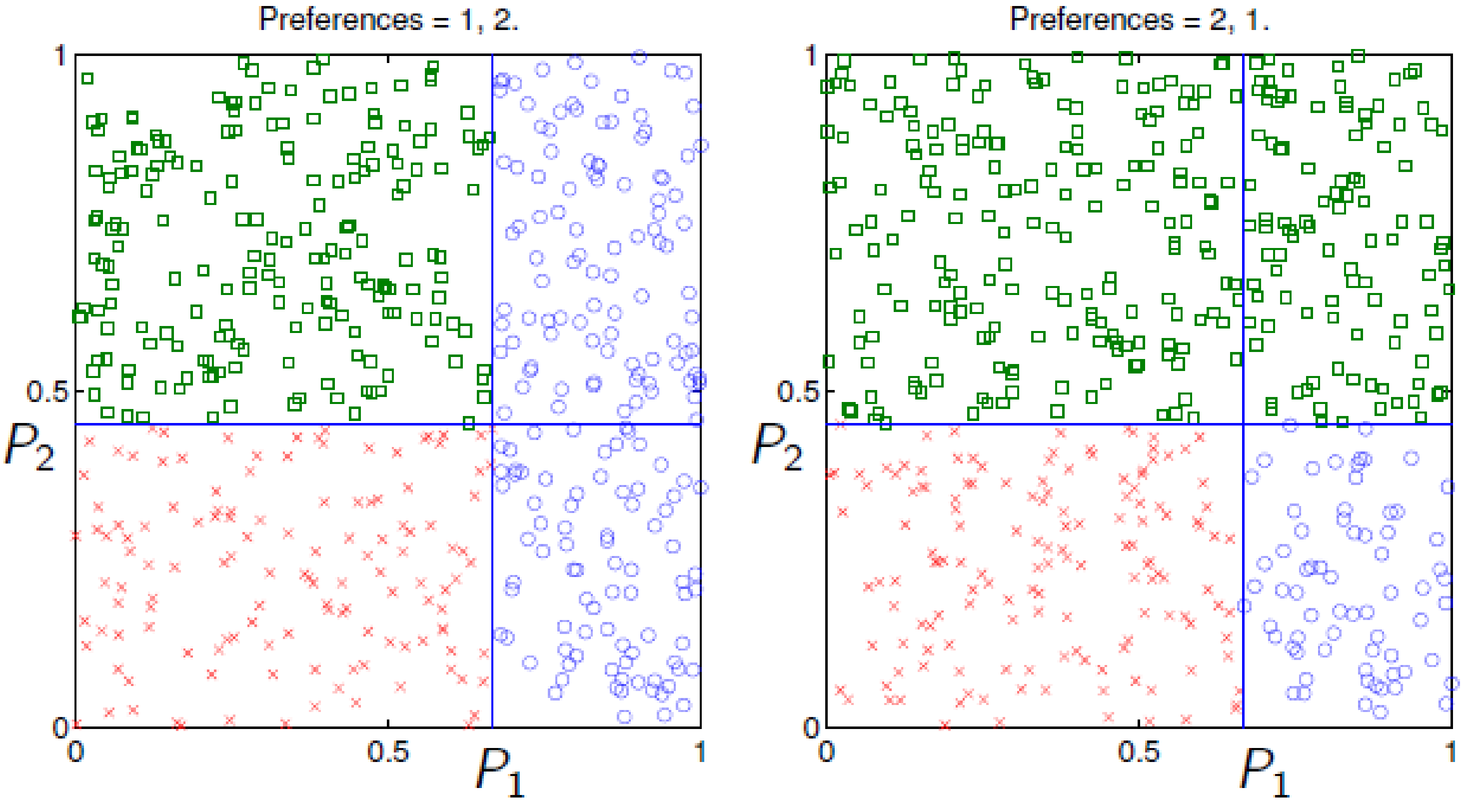
A Simpler description



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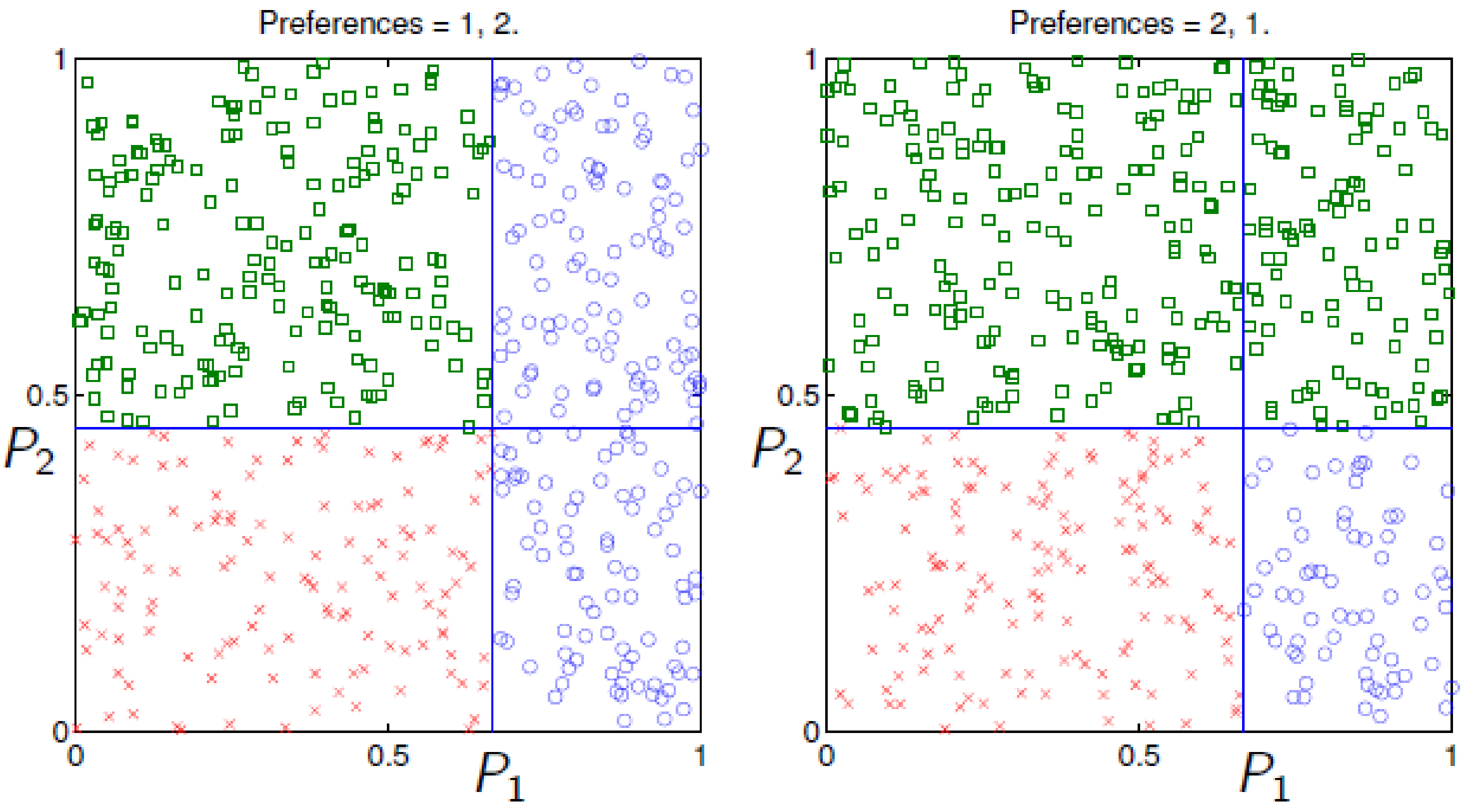
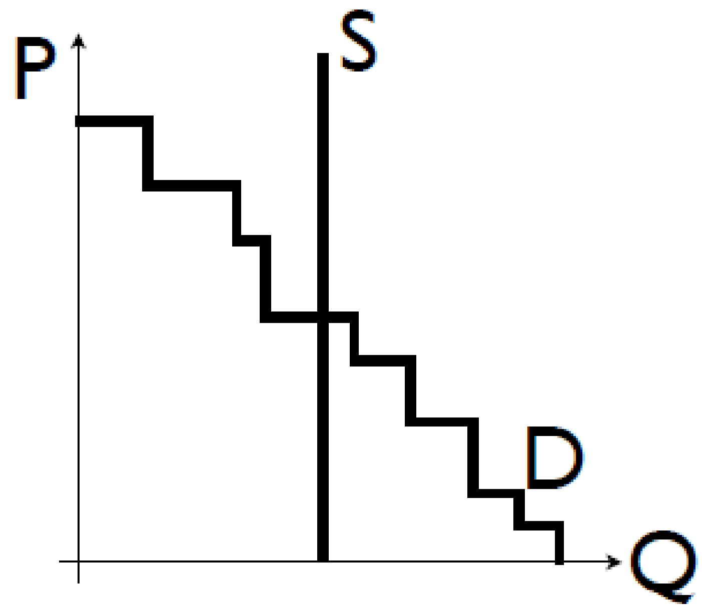


$$D(P) = S$$

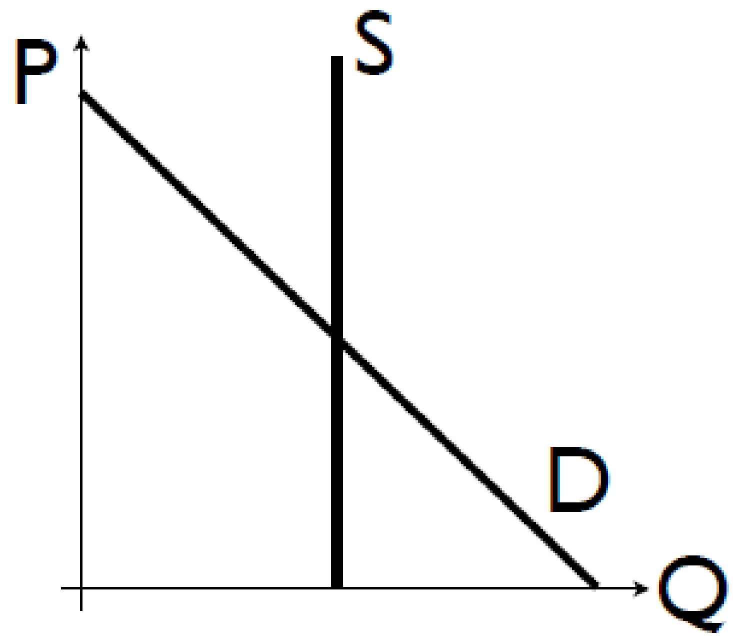
Two Simplifications

- Characterization of stable matching in terms of supply and demand



Two Simplifications

- Characterization of stable matching in terms of supply and demand
- Continuum of agents



Continuum Model

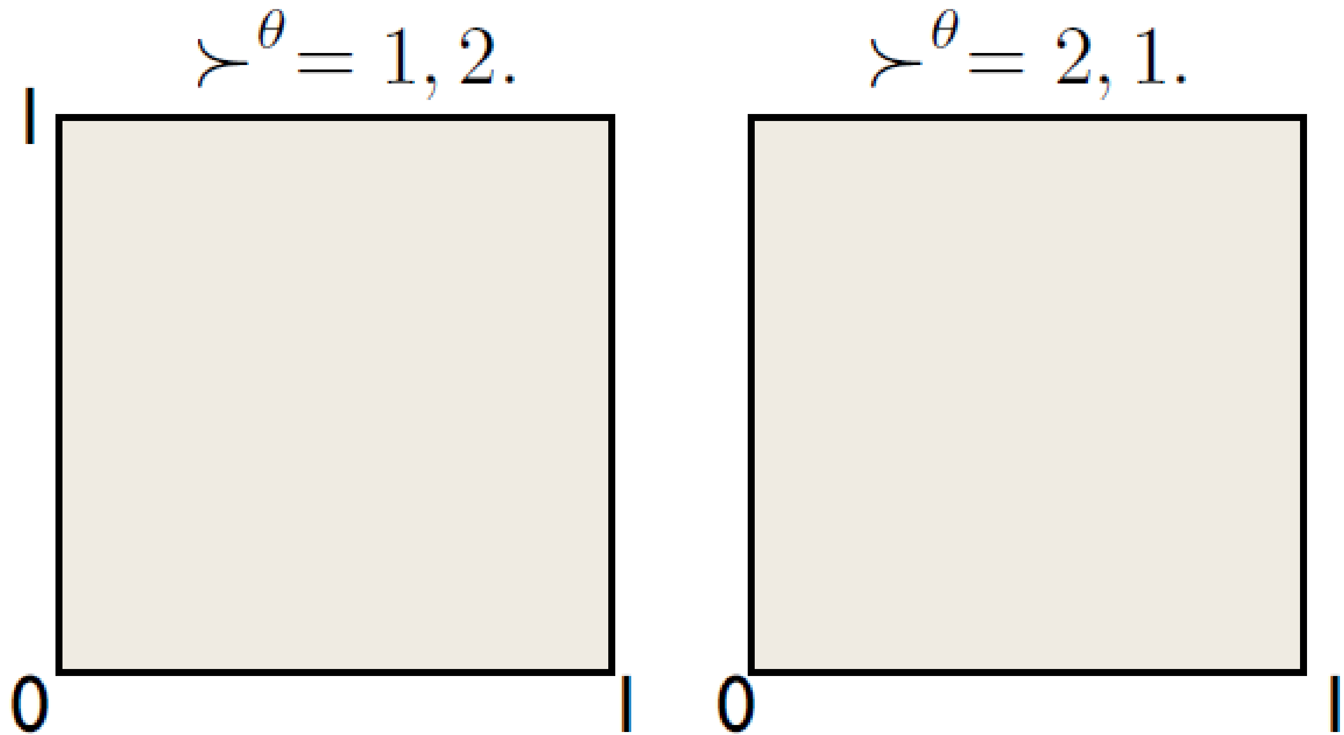
The Continuum Model

- Colleges $c \in \{1, \dots, C\}$
  - Capacities $S_c > 0$
- Student types $\theta = \left(\succ^\theta, e^\theta\right) \in \Theta$
  - $\succ^\theta$ is the student's strict preference over colleges
  - $0 \le e_c^\theta \le 1$ is the college c's score for the student, higher is more preferred

- An economy is $E = [\eta, S]$ where η is a distribution over student types
- **Assumption** (strict preferences): Colleges' indifference curves have measure 0

$$\eta\left(\left\{\theta \in \Theta \mid e_c^\theta = x\right\}\right) = 0 \quad \forall x, c$$

Example



$S_1 = .25, S_2 = .50$

η uniform

Matchings

- **Definition:** a matching is a function

$$\mu: C \cup \Theta \to 2^{\Theta} \cup (C \cup \Theta)$$

such that

1. Each student is matched to a college or itself
2. Each college is matched to a set of students such that $\eta\big(\mu(c)\big) \le S_c$
3. Consistent $\mu(c) \ni \theta \iff c \in \mu(\theta)$
4. Right continuous: For $\theta^k = \big(\succ, e^k\big), e^{k+1} \le e^k$ then $\mu\big(\lim \theta^k\big) = \lim \mu\big(\theta^k\big)$

Stable Matchings

- **Definition:** A matching is stable if it is not blocked by any college-student pair
  - Student θ prefers c over $\mu(\theta)$
  - c either:
    1. did not fill its quota $\eta\big(\mu(c)\big) < S_c$
    2. is matched to some $\theta' \in \mu(c)$ where $e_c^{\theta'} < e_c^{\theta}$

Supply and Demand

- A vector of cutoffs is a vector $P \in [0,1]^C$ specifying a minimal score (cutoff) P_c for each college

- The demand of student θ given cutoff P is her most preferred college that would accept her

$$D^\theta(P) = \arg\max_{\succ^\theta}\{c \mid P_c \leq e_c^\theta\}$$

- Aggregate demand $D(P)$ is the mass of students demanding each college

$$D_c(P) = \eta(\{\theta \mid D^\theta(P) = c\})$$

Market Clearing cutoffs

- **Definition:** P^* is a vector of market clearing cutoffs if for all c

$$D_c(P^*) \leq S_c$$

with equality if $P_c^* > 0$

Results

Relation between Matching and Cutoffs

- Given a stable matching μ let $P = \mathcal{P}\mu$ be the scores of the marginal accepted students

$$P_c = \inf_{\theta \in \mu(c)} e_c^\theta$$

- Given market clearing cutoffs P let $\mu = \mathcal{M}P$ be the match result from the demand under P

$$\mu(\theta) = D^\theta(P)$$

Cutoff Lemma

- **Lemma:** $\mathcal{M}$ *and* $\mathcal{P}$ *take stable matchings into market clearing cutoffs, and are inverses of each other*

Standard Results Carry

- The deferred acceptance algorithm converges to a stable matching
  - A stable matching exists
- The set of stable matchings is a lattice
  - The set of market clearing cutoffs is a lattice
- The measure of students match to each college is the same in all stable matchings (rural hospital theorem)

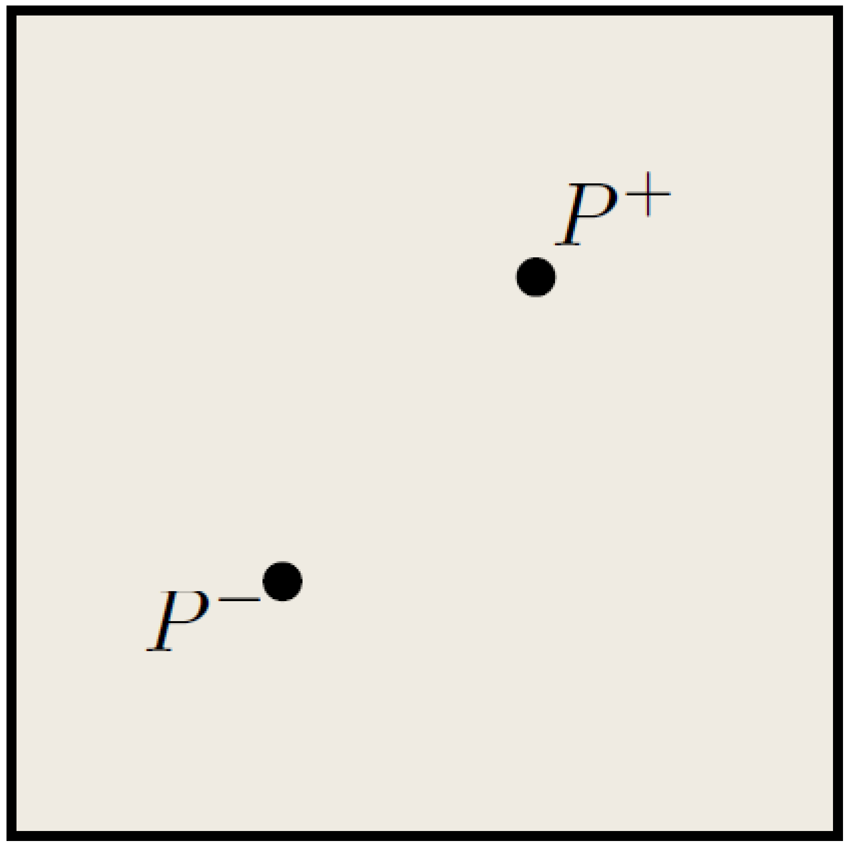
Uniqueness Theorem

Let $E = [\eta, S]$ *be a continuum economy*

I. *If* η *has full support then there is a unique stable matching*

II. *If* $D(\cdot)$ *is continuously differentiable then for almost any* S *there is a unique stable matching.*

Proof (I) – Full Support



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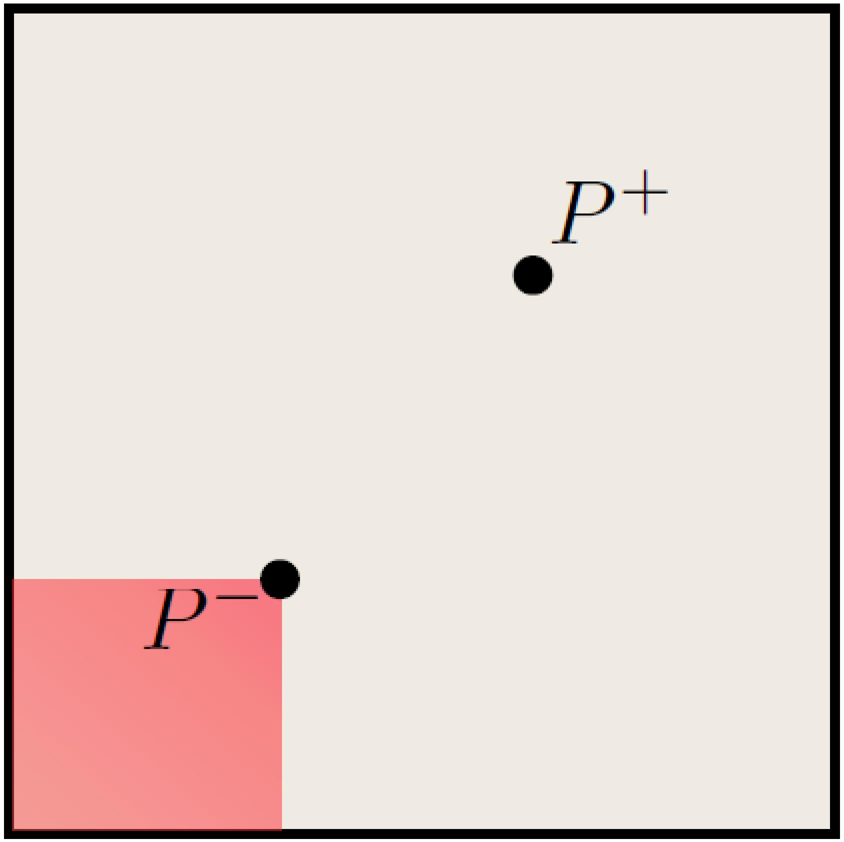

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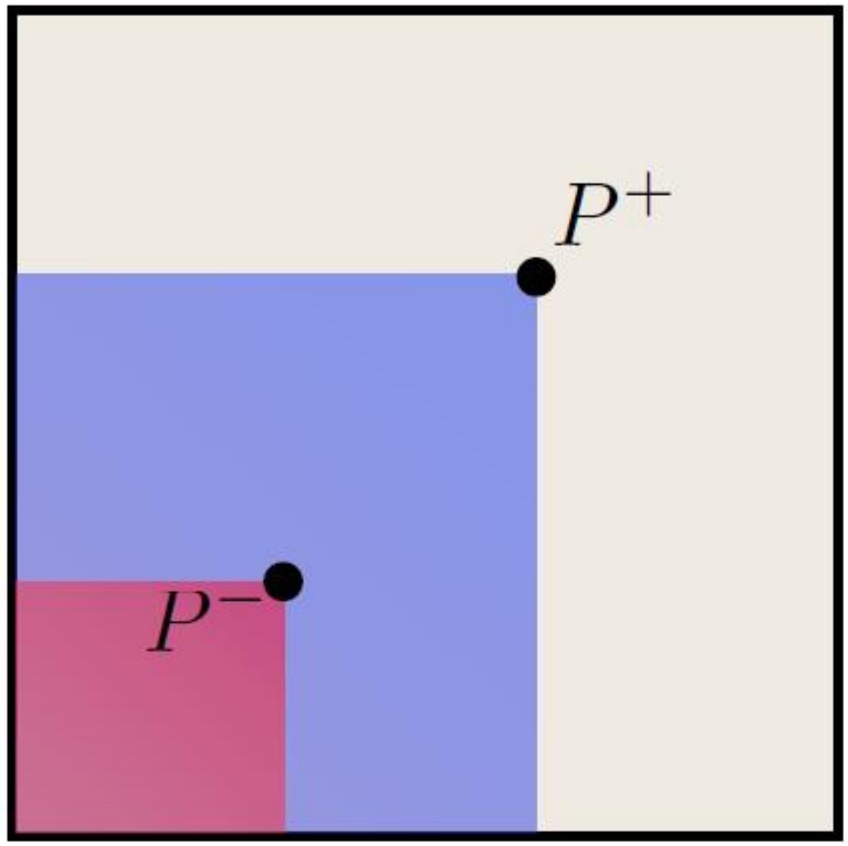
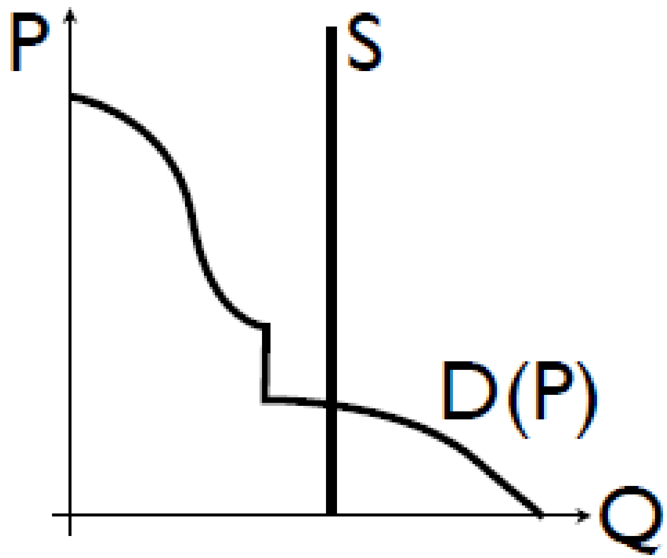


Proof (II) – Generic S

- Sard's Theorem: for almost every S, for all solutions P^* to $D(P) = S$ we have that the derivative $\partial D(P^*)$ is nonsingular.



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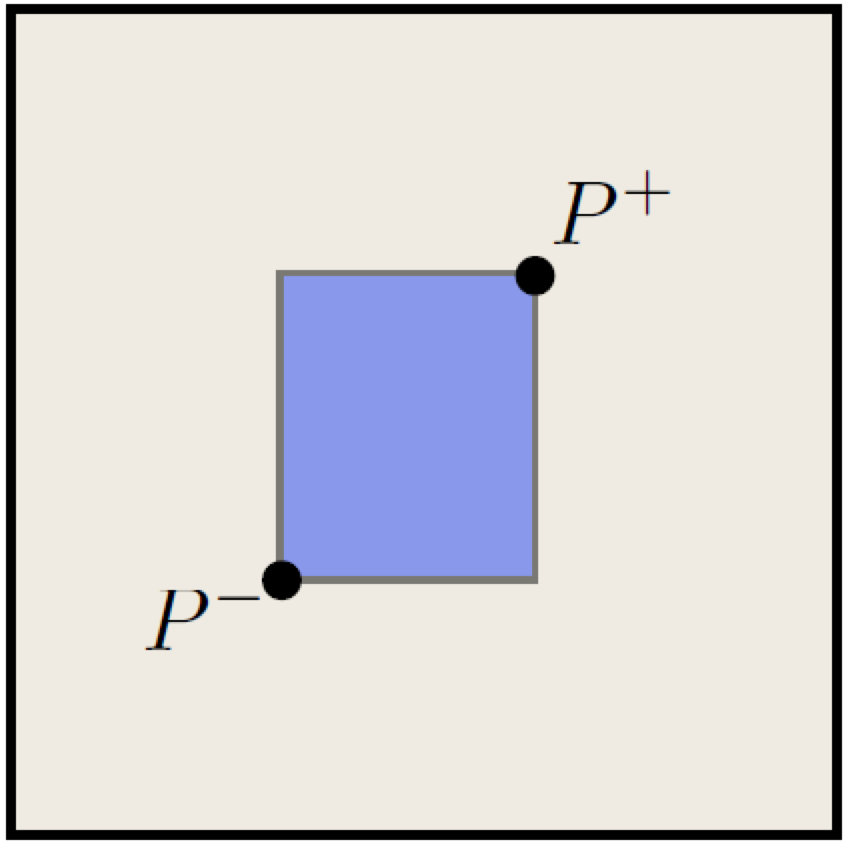


Proof (II) – Generic S

- The mass of unmatched workers

$$1 - \sum_c D_c(P)$$

  is constant in the cube $[P^-, P^+]$
- This implies the matrix $\partial D(P^-)$ is singular
- By Sard's Theorem, this is possible only for a measure 0 of S values.

Discrete Economies

- We can describe a discrete economy by $F = [\eta, S]$ where η is composed of a finite number of atoms
- **Lemma:** for a discrete economy $\mathcal{M}$ *and* $\mathcal{P}$ *take stable matchings into market clearing cutoffs, and* $\mathcal{MP}$ is the identity.

Convergence Notions

- E^k (or F^k) converges to E if
  - $\eta^k \to \eta$ in weak convergence of measures
  - $S^k \to S$
- Matchings $\mu^k \to \mu$ if $\mathcal{P}\mu^k \to \mathcal{P}\mu$
- The diameter of the set of stable matchings is

$$\sup |P' - P|$$

for any two market clearing cutoffs P, P'

Convergence Theorem

Let $E = [\eta, S]$ be an economy with a unique stable matching μ, then

I. Any sequence of stable matching of finite economies $F^k \to E$ converges to μ

II. The diameter of the set of stable matchings of F^k converges to 0

III. The stable matching correspondence is continuous at E

Applications

School Competition

- Does competition over students give schools incentive to invest in quality? (Tiebout, Hoxby in labor, Hatfield et al (2011) in matching)
  - Can we quantify the incentives?
  - What type of investment will be pursued?

School Competition – Setup

- We set up a continuum economy
- Set of students I
  - Scores distributed with continuous positive density
  - Student i utility for school c is $u_c^i(\delta_c)$, where δ_c is a quality measure for school c.
  - For quality δ there will be a distribution η_δ over student types
- Quality of entering class

$$Q_c(\delta) = \int_{\mu_\delta(c)} e_c^\theta \, d\eta_\delta(\theta)$$

Returns to investment in Quality

$$\frac{dQ_c}{d\delta_c} = \underbrace{[\bar{e}_c - P_c] \cdot N_c}_{\text{Direct Effect}} - \underbrace{\sum_{c' \neq c} [\bar{P}_{c'c} - P_c^*] \cdot M_{c'c} \cdot (-\frac{dP_{c'}^*}{d\delta_c})}_{\text{Market Power Effect}} .$$

- Direct effect of benefiting marginal students is ~ 0
- Indirect effect of benefiting marginal students
  - is ~ 0 when the market is thick (Hatfield et al)
  - Can be negative

Convergence of Random Economies

- Let $E = [\eta, S]$ have a unique stable matching μ which corresponds to the market clearing cutoff $P^* > 0, D(\cdot\,|\eta)$ is C^1 and $\partial D(P^*)$ is invertible.

- Let F^k be a finite economy generated by randomly and independently drawing k students from the distribution η and a capacity per student of $S^k \to S$ (that is, capacity is $\lfloor S^k \cdot k \rfloor$)
- Let $\tilde{\mu}^k$ be the random variable corresponding to a stable matching of F^k

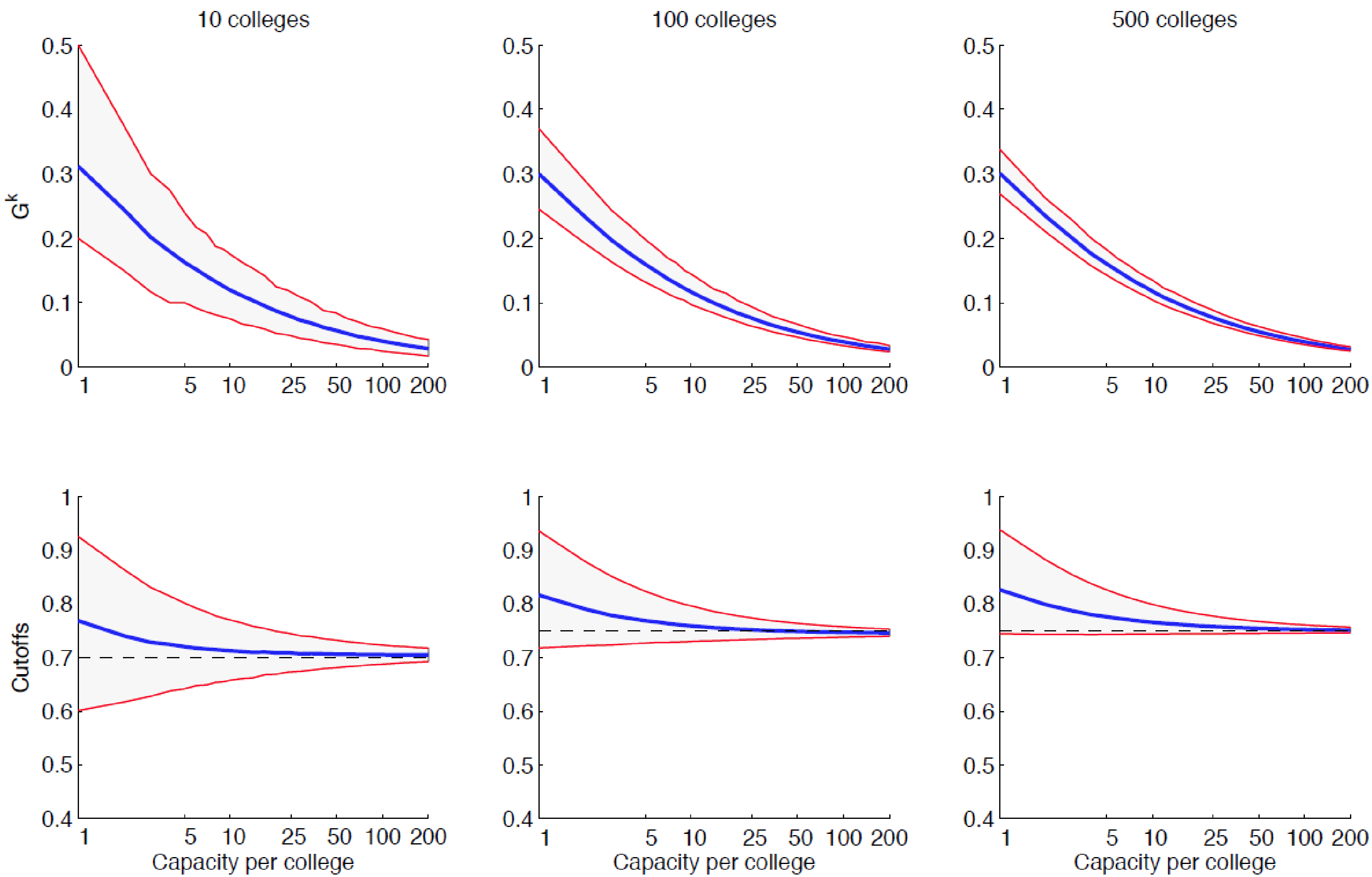
Convergence of Random Economies

Proposition:

I. *F^k converges almost surely to E and $\tilde{\mu}^k \to \mu$ almost surely*

II. *The convergence is "fast"*

Convergence Simulations



Convergence of Random Economies

- Gives convergence of RSD and PS (Che, Kojima 2010)
- Allows a tractable analysis of DA-STB, and comparison between DA-STB and DA-MTB

Conclusions

- New model of matching allowing for complex preferences but simple derivation
- The continuum model approximates large markets
- Solving for stable matching by solving supply-demand equations