Rigid Supersymmetry in Curved Superspace

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Work in progress with Thomas Dumitrescu, Nathan Seiberg.
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New handle to understand the dynamics of Susy theories.

Formulate new computable observables in known theories.

Many old and recent examples:

- $\mathcal{N} = 1$ on $AdS_4$ [Zumino;...]
- $\mathcal{N} = 1, 2, 4$ on $S^3 \times R$ [D. Sen; Romelsberger,...]
- $\mathcal{N} = 2$ on $S^2 \times R$ [Kim; Imamura, Yokoyama,...]
- $\mathcal{N} = 2$ on $S^4$ [Pestun;...]
- $\mathcal{N} = 2$ on $S^3$ [Kapustin, Willet, Yaakov;...]
Recent progress relied on case by case efforts.

Can we understand in generality the following:

- Which backgrounds allow for Susy?
- The corresponding Superalgebras.
- The Susy transformations of the fields.
- The structure of the Lagrangian.
A general treatment in terms of background superfields.

Backgrounds allowing 4 supercharges:

- $\text{AdS}_4$, $S^4$
- $S^3 \times R$, $S^3 \times S^1$

The partition function on $S^3 \times S^1$

Conclusions
The customary approach

- Start with a supersymmetric Lagrangian in flat space.
- Deform the metric to a curved manifold with size parametrized by $r$. Generically SUSY is broken.
- It is sometimes possible to preserve SUSY by deforming the SUSY variations of the fields and the Lagrangian order by order in $\frac{1}{r}$.

This procedure has several drawbacks:
- Case by case approach.
- Requires guesswork.
- Lots of algebra.
- The structure of the resulting theory is difficult to understand.
Consider an off-shell formulation of Supergravity and give arbitrary background values to the fields in the gravity multiplet:

- The metric $g_{\mu\nu}$
- Various auxiliary fields.
- Set the gravitino $\psi_{\mu\alpha} = 0$

**The rigid limit.**

Send $M_p \to \infty$ keeping the background values for the metric and auxiliary fields fixed. The auxiliary fields scale with weight one.

Some supersymmetry is preserved if it is possible to find $\zeta_{\alpha}$ such that the SUSY variation of the gravitino is zero:

$$\delta_\zeta \psi_{\mu\alpha} = 0 \Rightarrow \nabla_\mu \zeta_{\alpha} = M_{\mu\alpha}^{\beta} \zeta_{\beta}$$

where $M_{\mu}$ depends on the metric and auxiliary fields.
Great simplification.

Many terms in the Sugra Lagrangian drop out.

Different than Linearized Sugra.

We do not impose e.o.m. for the auxiliary fields. Different of shell formulations of SUGRA lead to distinct deformations.
At order $1/r$ in the expansion in the size $r$ of the manifold the auxiliary fields couple linearly to definite components of the Supercurrent.

At this order the deformation of the flat space theory can be described also when a Lagrangian is not available.

At order $1/r^2$ there are curvature terms and terms quadratic in the auxiliary fields.

The expansion in the size of the manifold ends at order $1/r^2$.

More explicit details to follow....
The Rigid Limit of SUGRA: Comments

\[ \nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu \alpha}^{\ \beta} \zeta_\beta \]

- The "Killing" equation for \( \zeta \) depends only on the fields in the gravity multiplet through \( \nabla_\mu \) and \( \mathcal{M}_\mu \).
- We do not need to find backgrounds for matter and Sugra satisfying the e.o.m.

\[ \Downarrow \]

There is no dependence on the matter content.

Generalized treatment of different theories.
Auxiliary fields: $b_\mu$ a real vector and $M, \overline{M}$ a complex scalar. Appropriate to theories with a Ferrara-Zumino supercurrent:

$$\bar{D}^{\dot{\alpha}} J_{\alpha \dot{\alpha}} = D_\alpha X ; \quad \bar{D}_{\dot{\alpha}} X = 0.$$ 

$b_\mu$ couples to the lowest component of $J_\mu$ while $M$ couples to the lowest component of $X$.

Imposing that the Susy variation of the gravitino vanishes we get:

$$\nabla_\mu \zeta = \frac{i}{6} \left( M \sigma_{\mu} \zeta + 2 b_\mu \zeta + 2 b^\nu \sigma_{\mu \nu} \zeta \right)$$

$$\nabla_\mu \bar{\zeta} = -\frac{i}{6} \left( -\bar{M} \bar{\sigma}_{\mu} \zeta + 2 b_\mu \bar{\zeta} + 2 b^\nu \bar{\sigma}_{\mu \nu} \bar{\zeta} \right)$$

In Euclidean space $-$ is not c.c. and we can allow $M, \overline{M}$ and $b_\mu$ to take arbitrary complex values.
Example: the bosonic terms for a Wess-Zumino model

\[
\frac{1}{e} \mathcal{L}_0 = K_{ij} \left( F^i \bar{F}^j - \partial_\mu \bar{\phi}^j \partial^{\mu} \phi^i \right) + F^i W_i + \bar{F}^j \bar{W}_j
\]

The terms linear in the auxiliary fields are:

\[
\frac{1}{e} \mathcal{L}_1 = - \left( \frac{1}{3} K_i F^i + W \right) M - \left( \frac{1}{3} K_i \bar{F}^i + W \right) \bar{M} - \frac{i}{3} b^\mu \left( K_i \partial_\mu \phi^i - K_i \partial_\mu \bar{\phi}^i \right)
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
\overline{X} M \quad \overline{X} \bar{M} \quad b^\mu j^\mu_{FZ}
\]

At order \( \frac{1}{r^2} \) we get:

\[
\frac{1}{e} \mathcal{L}_2 = \left( \frac{1}{6} \mathcal{R} + \frac{1}{9} M \bar{M} - \frac{1}{9} b_\mu b^\mu \right) K
\]
The Susy transformation of the fields are deformed from their flat space counterparts:

\[
\delta \phi^i = -\sqrt{2} \zeta \psi^i \\
\delta \psi^i_\alpha = -\sqrt{2} \zeta_\alpha F^i - i \sqrt{2} (\sigma^\mu \zeta)_{\alpha} \partial_\mu \phi^i \\
\delta F^i = -i \sqrt{2} \overline{\zeta} \overline{\sigma}^\mu \nabla_\mu \psi^i - \frac{\sqrt{2}}{3} \bar{M} \zeta \psi^i + \frac{\sqrt{2}}{6} b_\mu \zeta \overline{\sigma}^\mu \psi^i
\]
By requiring that $\delta_\zeta \psi = \delta_\zeta \bar{\psi} = 0$ for four independent $(\zeta, \bar{\zeta})$ we obtain the following integrability conditions:

\begin{align*}
    b_\mu &= 0 \\
    \partial_\mu M &= \partial_\mu \bar{M} = 0 \\
    W_{\mu\nu\kappa\lambda} &= 0 \\
    \mathcal{R}_{\mu\nu} &= \frac{1}{3} g_{\mu\nu} M \bar{M}
\end{align*}

\[\Downarrow\]

AdS$_4$, $S^4$

\[\Downarrow\]

Conformal flatness

OR

\begin{align*}
    M &= \bar{M} = 0 \\
    \nabla_\mu b_\nu &= 0 \\
    W_{\mu\nu\kappa\lambda} &= 0 \\
    \mathcal{R}_{\mu\nu} &= -\frac{2}{9} (b_\mu b_\nu - g_{\mu\nu} b_\rho b^\rho)
\end{align*}

\[\Downarrow\]

AdS$_3 \times R$, $S^3 \times R$, PP wave
AdS$_4$ is obtained for $M = \bar{M} = \frac{3}{r}$ and $b_\mu = 0$.

$S^4$ can be obtained from Euclidean AdS$_4$ by relaxing $M^* = \bar{M}$.

$$M = \bar{M} = \frac{3}{ir} \quad b_\mu = 0$$

- The Superalgebra is $OSp(1|4)$
- By Kahler transformations $W$ can be adsorbed in $K$:

  $$K \rightarrow K + Y(\phi) + \bar{Y}(\bar{\phi})$$
  $$W \rightarrow W + \frac{1}{3}MY, \quad \bar{W} \rightarrow \bar{W} + \frac{1}{3}\bar{M}\bar{Y}$$

  The holomorphic data is not protected.

- Turning on $M, \bar{M}$ explicitly breaks any U(1) R-symmetry.
- For Superconformal theories all $1/r$ terms vanish.
For $S^4$ the auxiliary fields do not have the standard reality; as a consequence the Lagrangian is not reflection positive.

The problematic terms are those linear in $M, \overline{M}$. For a non-conformal theory with mass scale $m$ these are $O\left(\frac{m}{r}\right)$.

This is consistent with there being no unitary SUSY theory in dS.

On $S^4$ the isometry group $SO(5)$ in $OSp(1|4)$ does not have a compact real form.

- The commutator of two supercharges is a complexified isometry.
- Hard to localize (possible in $\mathcal{N} = 2$ with superalgebra $OSp(2|4)$ [Pestun...])
The (Euclidean) cylinder $S^3 \times R$ is obtained by setting

$$b_\mu = \frac{i}{r} \delta^4_\mu, \quad M = \bar{M} = 0$$

The Supersymmetry algebra is $SU(2|1) \times SU(2)$

In pure Old Minimal Sugra ($H$ generates translations along $R$):

$$[Q_\alpha, H] = Q_\alpha$$

It is therefore not possible to compactify to $S^3 \times S^1$
For a theory with a \( U(1) \) R-symmetry we can turn on an imaginary background for a \( U(1)_R \) connection \( A_\mu \) along the cylinder’s axis. This is equivalent to redefining \( H' = H + R \)

\[
\left[ Q_\alpha, H' \right] = 0, \quad SU(2|1) \times SU(2) \times U(1)
\]

It is then possible to compactify on \( S^3 \times S^1 \)

If additional \( U(1)_f \) flavor symmetries are present we can add complex background gauge fields \( A^f_\mu \) along \( S^1 \).

The dependence on \( A^f_\mu \) is holomorphic.
Alternatively we can work in New Minimal Sugra which naturally couples to theories with an R-symmetry.

The auxiliary fields are a two form $B_{\mu\nu} \sim B_{\mu\nu} + \partial[\mu a_{\nu}]$ and $A_{\mu} \sim A_{\mu} + \partial_{\mu}\phi$.

These couple to components of the "R-multiplet" a supercurrent multiplet having the conserved R-current as the lowest component and satisfying:

\[ \bar{D}^{\dot{\alpha}} R_{\alpha\dot{\alpha}} = \chi_{\alpha} \quad ; \quad \bar{D}_{\dot{\alpha}} \chi_{\alpha} = 0 \quad ; \quad D^{\alpha} \chi_{\alpha} = \bar{D}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \]

The "Killing" spinor eq is:

\[ \nabla_{\mu} \zeta = -i V^{\nu} (\sigma_{\mu\nu} \zeta) - i (V_{\mu} - A_{\mu}) \zeta, \quad V = * d B \]
The cylinder $S^3 \times R$ is obtained by turning on $H = dB$ threading $S^3$ and constant $A_\mu$ along the axis.

Different values of $A_\mu$ correspond to redefinitions of $H$

- For $A_\mu = 0$ we recover the old minimal case:
  \[
  \{ Q_\alpha, \bar{Q}_\beta \} = 2\delta_{\alpha\beta} H + 2\sigma^i_{\alpha\beta} J^i_l \quad [H, Q_\alpha] = Q_\alpha .
  \]

- For $A_\mu = V_\mu$ where $V = \ast H$ we can compactify to $S^3 \times S^1$
  \[
  \{ Q_\alpha, \bar{Q}_\beta \} = 2\delta_{\alpha\beta} (H - R) + 2\sigma^i_{\alpha\beta} J^i_l \quad [H, Q_\alpha] = 0 .
  \]

- For $A_\mu = \frac{3}{2} V_\mu$ we get:
  \[
  \{ Q_\alpha, \bar{Q}_\beta \} = 2\delta_{\alpha\beta} \left( H - \frac{3}{2} R \right) + 2\sigma^i_{\alpha\beta} J^i_l \quad [H, Q_\alpha] = -\frac{1}{2} Q_\alpha .
  \]

For a superconformal theory $H$ can then be identified with the dilatation operator.
From $S^3 \times S^1$ By dimensional reduction along the $S^1$ direction we obtain a $\mathcal{N} = 2$ theory on $S^3$ with U(1) R-symmetry.

- The superalgebra is $SU(2|1) \times SU(2)$
- As for the theory on $S^4$ the theory in not reflection positive unless superconformal.
- $\text{Im} A^f_\mu$ leads to shifts in the choice of $U(1)_R$ charges.
- $\text{Re} A^f_\mu$ leads to real masses in 3d.
- New terms can be added to the 3d theory (e.g. Chern-Simons)
Trace over the Hilbert space. The $A^f_{\mu} = -i r \mu_f \delta^4_{\mu}$ are complex chemical potentials for the $U(1)_f$.

$$Z = Tr(-1)^F \exp \left( -\frac{\beta}{r} H - \frac{\beta}{r} \sum_f \mu_f Q_f \right)$$

- Gets contributions only from short representations of $SU(2|1)$. As reps of the bosonic subalgebra $SU(2) \times U(1)$ these are $(j, 2j)$.

$$\{ Q_\alpha, \bar{Q}_\beta \} = 2 \delta_{\alpha\beta} (H - R) + ...$$

The $U(1)$ in $SU(2|1)$ is $H - R \Rightarrow$ on short reps $H = 2j + R$.

- The values of $H$ are quantized. $Z$ is independent of small deformations of the lagrangian. It is the same in the UV and IR.
Free field computations in the UV are possible [Romelsberger].

Used to test dual descriptions in the IR. [Romelsberger; Dolan, Osborn; Spiridonov, Vartanov; ...]

For superconformal theories it reduces to the superconformal index [Kinney, Maldacena, Minwalla, Raju]

The dependence on $\mu_f$ is holomorphic
Less than 4 supercharges

The conditions for four supercharges are very restrictive:

- Conformal flatness
- In Old Minimal either $b_\mu = 0$ or $M = \overline{M} = 0$
- Constant $M$ or covariantly constant $b_\mu$

We would like to understand how do spacetimes allowing for less supersymmetry look like.

Extend to different number of dimensions
Turning on background values for the fields in the supergravity multiplet and taking the rigid limit allows a general description of rigid SUSY in curved superspace.

The \((\mathcal{M}, g)\) allowing for SUSY can be identified independently from the matter content.

Requiring 4 supercharges only allows a restricted set of \((\mathcal{M}, g)\). The properties of theories formulated on these spaces are more transparent than in the traditional approach.

Requiring less than 4 supercharges a richer set of \((\mathcal{M}, g)\) is allowed. Hopefully this will provide new tools to understand the dynamics of Susy field theories.
Thank You!