Matching with Contracts

Paul Milgrom Lecture

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Thanks to John Hatfield for sharing his slides, on which these slides rely

Themes

- "Matching with Contracts" nests various matching models and auction models (including Vickrey), some with and some without money transfers or other contract terms.
- The Gale-Shapley algorithm is a planning algorithms, related to
  - *Walrasian price-adjustments,* which can be mimicked in models without prices, but…
  - *Marshallian quantity adjustments*, which will be part of the basis of the FCC's "incentive auction," and…
  - *Cumulative offer algorithms*, which apply to both matching models and "combinatorial" auctions.
- Mathematical connections are particularly tight among models satisfying a suitably general "substitutes" condition.

Literature

- This lecture is based mainly on Hatfield and Milgrom (2005), which is part of the large matching theory literature.

- Kelso and Crawford (1982): Introduced prices into the two-sided matching framework.
- Ausubel and Milgrom (2002): Introduced cumulative offer algorithms to address auction problems without market clearing prices.

Contract Environment

- Our matching environment is characterized by three finite sets
  - D is the set of "doctors"
  - H is the set of "hospitals"
  - $F = D \cup H$ is the set of market participants
  - T is the set of "terms" (wages, location, hours, etc)

- A contract is an element of the finite set $X \subseteq D \times H \times T$.

Choice Functions

- These market participants' preferences are encapsulated in choice functions $C_f(Y)$. These are restricted so that
  - $C_d(Y) \subseteq \{x \in Y | x_D = d\}$
    A doctor chooses among the contracts bearing her name
  - $C_h(Y) \subseteq \{x \in Y | x_H = h\}$
    A hospital chooses among the contracts bearing its name
  - Assume that there are also associated preference relations P_f
- For the market as a whole, the doctors' and hospitals' choices from their "budget sets" are denoted by
  - $C_D(Y) = \cup_{d \in D} C_d(Y)$
  - $C_H(Y) = \cup_{h \in H} C_h(Y)$

Matching with Contracts

SUBSTITUTES

Universal Substitution

- We want to define the concept of substitution to be "universal" in two senses.
  - Whether x and y are substitutes does not depend on the presence of other alternatives.
  - Every contract is a substitute for every other contract.

- **Definitions and Notation**.
  1. Contracts are ***substitutes*** if for all $x,y \in X$ and $Z \subseteq X$, if $x \notin C_f(Z \cup \{x\})$ then $x \notin C_f(Z \cup \{x,y\})$.
  2. A function $F:X \to Z$ is ***isotone*** if $x \geq y \Rightarrow F(x) \geq F(y)$. (Informally, isotone means non-decreasing).
  3. The ***rejection function*** is $R_f(Z) \equiv Z - C_f(Z)$.

- **Proposition**. Contracts are substitutes for f if and only R_f is isotone.

Equivalent Definitions

- The matching with contracts model embeds traditional matching models.
  - Taking T to be a set of wages, D×H×T is the Kelso-Crawford model.
  - Taking T to be a singleton, we can identify X with D×H to get the Gale-Shapley algorithm.

- Equivalences
  - Contracts are substitutes in the Gale-Shapley model if and only if preferences are substitutable in the sense of Roth & Sotomayor.
  - Contracts are substitutes in the Kelso-Crawford model if and only if they are price-theoretic "gross substitutes."
    - Key step: raising a worker's ("required") wage means reducing the set of options available to the firm.
    - Concept: prices identify the firm's choice set.

Examples of Substitutes

1. "Responsive preferences" (additively separable values).
2. Separate openings for men and women.
3. Hospital openings with "reversion."
4. A hockey team regard players as substitutes in the case where
   - the team needs a goalie, a defenseman, and a forward…
   - the team's payoff is the sum of the values of the players in the positions to which they are assigned
   - the team can assign any of its players to any position
5. …

Dual Characterization, 1

- In the classical theory of the firm, let w be a vector of wages and let π be the firm's indirect profit function.
  - $\pi(w) = \max_{x \in S} f(x) - w \cdot x$.
  - By Hotelling's lemma, for almost all wage vectors w (points of differentiability of π), the firm's demand for workers is $x^* = -d\pi/dw$.
  - So, in the theory of the firm,
    workers are substitutes if and only π is submodular (roughly: negative mixed partial derivative).

Dual Characterization, 2

- **Definition**: The (indirect utility) function U_f represents the preference/choice relation P_f /C_f if

$$U_f(Y) > U_f(Z) \Leftrightarrow C_f(Y)\ P_f\ C_f(Z)$$

- **Definition**. U is ***submodular*** if for all $x \in X$ and $Y \subseteq Z \subseteq X$, $U(Y \cup \{x\}) - U(Y) \geq U(Z \cup \{x\}) - U(Z)$.

- **Theorem (Hatfield & Kominers)**. Contracts are substitutes for participant f if and only if the preferences of f can be represented by a submodular indirect utility function.

Proof.

- Suppose that that contracts are NOT substitutes for f.
- Then, there exists a set $Z \subseteq X$ and contracts $x,y \in X$ such that $x \notin C_f(Z \cup \{x\})$ and $x \in C_f(Z \cup \{x,y\})$. By strict preference, this implies that $C_f(Z \cup \{x,y\}) \succ_f C_f(Z \cup \{x\}) = C_f(Z)$.
- Suppose that indirect utility function U_f represents $\succ_f$. Then, $U_f(Z \cup \{x,y\}) > U_f(Z \cup \{x\}) = U_f(Z)$. So, $U_f(Z \cup \{x,y\}) - U_f(Z \cup \{x\}) > 0 = U_f(Z \cup \{x\}) - U_f(Z)$, in contradiction to submodularity.
- If contracts are substitutes, then the indirect utility function that assigns utility $1 - 2^{-n}$ to the n^{th} most preferred set of contracts is submodular and represents preferences.

Matching with Contracts

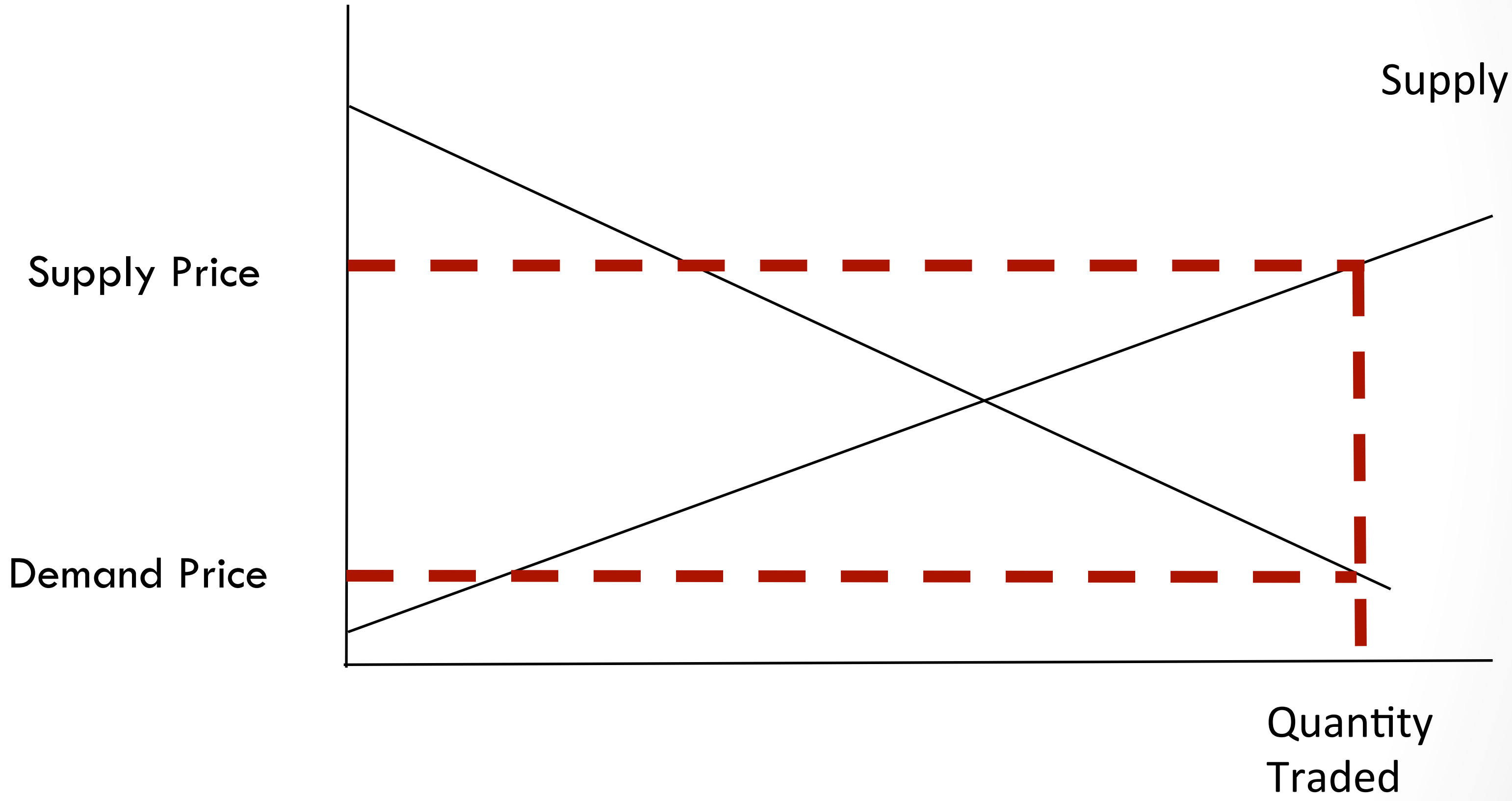
STABILITY

Stability Defined

- A set of contracts $Y \subseteq X$ is ***stable*** if it satisfies two conditions:
  1. *Individual rationality*: for all $f \in F$, $C_f(Y)=Y_f$.
  2. *Unblocked*: There exists no set of contracts Z such that $Z-Y \neq \varnothing$ and for all $f \in D \cup H$, $Z_f \subseteq C_f(Y \cup Z)$.

- Our next task is to characterize stability in terms of agents making optimal choices from a “budget set,” and those sets being mutually consistent.
- Then, we look at an algorithm to find stable allocations and characterize it as related to both the Gale-Shapley algorithm and to Marshallian dynamics.

Marshallian Dynamics



Generalized Deferred Acceptance

- Informally, the state of the algorithm at any moment is a pair (X_D,X_H) representing the contracting opportunities as seen by the doctors and hospitals, respectively.
  - Formally, we require $X_D \subseteq X$ and $X_H \subseteq X$.
- As the algorithm progresses, the doctors' "budget set" consists of all those contracts that the hospitals have not just rejected, and the hospitals consider… [symmetrically]

$$F_D(X_H) = X - R_H(X_H)$$

$$F_H(X_D) = X - R_D(X_D), \text{ and}$$

$$F(X_H,X_D) = (F_D(X_H),F_H(X_D))$$

Substitutes and Stability

- **Theorem**. If contracts are substitutes for all market participants, then $A \subseteq X$ is a stable allocation if and only if there exists a fixed point $(X_D, X_H) = F(X_D, X_H)$, such that $A = X_D \cap X_H$.

- Fixed point:

  $X_D = X - R_H(X_H)$ = "doctors' budget sets"

  $X_H = X - R_D(X_D)$ = "hospitals' budget sets"

  Note: $X_D \cup X_H = X$

Proof Sketch

- If $A=X_D \cap X_H$, then…
  - Any individually unacceptable contract is always rejected and no such contract is included from A.
  - By construction, contracts not in $X_D \cap X_H$ would, if proposed, be rejected by one side or the other.
  - By substitutes, if multiple additional contracts were proposed, they would still be rejected (because the rejection functions are isotone).
- If A is stable…
  - then, we can construct X_D and X_H as follows: every contract not in A would, if proposed, be rejected by its h or d, so it can be added to X_H or X_D, respectively.

Bounded monotone sequences converge

ASIDE ON LATTICE MATHEMATICS

Lattices and Order

- Let ≥ be a transitive, reflexive relation on a set X.
- **Definition**. (X,≥) is a ***lattice*** if for all x,y∈X, inf{x,y}∈X and sup{x,y}∈X both exist.
- **Notation**. $x \wedge y = \inf\{x,y\}$ and $x \vee y = \sup\{x,y\}$.
- Examples
  - X comprises the subsets of some base set S and $x \leq y$ means $x \subseteq y$.
    - $x \wedge y = x \cap y$ and $x \vee y = x \cup y$.
  - $X = \mathbf{R}^N$ and $x \geq y$ means $x_i \geq y_i$ for i=1,…,n.
    - $(x \wedge y)_i = \min(x_i, y_i)$ and $(x \vee y)_i = \max(x_i, y_i)$.
    - $\mathbf{R}^N$ is an example of a ***product lattice***.

Tarski's Fixed Point Theorem

- **Definitions**.
  1. A lattice $(X,\geq)$ is ***complete*** if for every non-empty subset $S\subseteq X$, $\inf(S)\in X$ and $\sup(S)\in X$.
     - Roughly, bounded monotone sequences have their limits in the set.
  2. A function $F:X\rightarrow X$ is ***isotone*** if $x\geq y \Rightarrow F(x)\geq F(y)$.

- **Theorem**.
  - If $(X,\geq)$ is a complete lattice and $F:X\rightarrow X$ is isotone, then the set of fixed points of F is a non-empty lattice.
  - Moreover, if X is finite, with largest element x_{max} and smallest element x_{min}, then the sequences $\{F^n(x_{max})\}$ and $\{F^n(x_{min})\}$ converge in finitely many steps to the largest and smallest fixed points of F, respectively.

LATTICE STRUCTURE IN MATCHING

Existence, Computation

- **Definition**: $(X_D,X_H) \geq (Y_D,Y_H)$ if both (1) $X_D \subseteq Y_D$ and (2) $Y_H \subseteq X_H$.
- With this definition,
  - higher pairs represent expanded opportunities for hospitals and contracted opportunities for doctors.
  - the domain of the function F described earlier is a *finite lattice*.
    - Meets and joins correspond to unions and intersections in the first component and to the reverse (intersections and unions) in the second component.
  - the operator F is isotone
- **Theorem**. The set of fixed points of the function F is non-empty and has a maximal element and a minimal element. Iterated applications of F starting from the highest (lowest) point in the lattice converges monotonically down (up) to the highest (lowest) fixed point.

Relation to Gale & Shapley

- The highest point in the lattice is the pair $(X_D,X_H)=(\varnothing,X)$, at which doctors have no opportunities and hospitals have all options still open.
- Iterated application of F
  - At the first application, hospitals reject all but their most preferred acceptable contracts, so that the next pair of contracts includes some offers to doctors.
  - At the second round, doctors reject all but their most preferred acceptable contracts from among the offered contracts, reducing the set of opportunities for hospitals.
  - …
- That the extremal fixed points are doctor-best and hospital-best follows from a comparison of the opportunity sets.
- Similarly, in a competitive model with substitutes, there is a seller-best point with the highest prices for all goods, and a buyer-best point with the lowest prices.
- Notice… as-if cumulative offers…

INCENTIVES AND RELATED

Law of Aggregate Demand

- **Definition**. The preferences of individual f satisfy the law of aggregate demand if for all $Y \subseteq Z \subseteq X$, $|C_f(Y)| \leq |C_f(Z)|$.
- In the price-theoretic version of the problem, an expanding opportunity set for the firm corresponds to lower wages for all workers.
  - The definition requires that when a firm is faced with lower wages, it should hire more workers.

Rural Hospitals Theorem

- **Theorem**. If each participant regards contracts as substitutes and its preferences satisfy the law of aggregate demand, then each "signs" the same number of contracts at every stable allocation.
- **Proof**. At the doctor-optimal stable allocation, the doctors' allocation coincide with their choices from the largest opportunity set, so each signs at least as many contracts as in any stable allocation. So, the total number of contracts signed is at least as large as at any stable allocation.
- Similarly, each hospital has its smallest opportunity set, and hence signs the smallest number of contracts, so the total number is at least as small…
- But each contract has one doctor and one hospital, so these numbers are equal to each other, and so equal to the corresponding numbers at any stable allocation.

Strategy-Proofness

- **<u>Theorem (Hatfield & Milgrom)</u>**. Suppose that each doctor can sign only one contract and that each hospital reports preferences satisfying the law of aggregate demand and for which contracts are substitutes. Then, the induced one-sided reporting game among doctors is strategy-proof.
  - Truth-telling is always optimal for doctors.

Proof Sketch

- The proof proceeds by showing that if any doctor d reports any preference P that leads to outcome x, then each of the following variations leads her to a weakly better outcome than the preceding one.
  1. Report preference P (leading to a set A, including x for d).
  2. Report that only x is acceptable.
     - Fewer allocations are blocked, so A is still stable. By LOAD, d is assigned at every stable allocation, so it must be get contract x.
  3. Report that contracts truly worse than x are unacceptable, but otherwise report preferences truthfully.
     - By isotonicity of F, the highest stable set is at least as good for doctors as A, so d must do at least as well.
  4. Report preferences truthfully.
     - Outcome unchanged from report #3.

Group-Strategy Proofness

- **<u>Theorem (Hatfield & Kojima)</u>**. Suppose that each doctor can sign only one contract and that each hospital reports preferences satisfying the law of aggregate demand and according to which contracts are substitutes. Then, the induced reporting game among doctors is weakly group strategy-proof. (No coalition of doctors can deviate in a way that is strictly beneficial to all of them.)

A First Lemma

- Lemma. Suppose that Z is the doctor-optimal stable allocation under preference profile P, in which doctor d gets contract z and suppose that y is preferred to z under P_d. Let P′ be the preference profile in which d's preference only is replaced by the one in which y is moved to the top of d's otherwise unchanged list. Then the doctor best stable match for P′ is also Z.
- Proof. Z is still stable, since it is still acceptable and the potential blocking pairs are unchanged.
- Suppose that the resulting allocation is Y. Since the mechanism is individually strategy-proof, $y \notin Y$. But then Y is blocked by whatever set had blocked coalition Y under P.

Proof

- Let S={1,…,n} be a minimal coalition that can, for some preferences, benefit by a joint deviation, changing its outcome from the result x of truthful reporting to some preferred outcome y.
- By the lemma, if n alone deviates, reporting the preferences $y_n \succ x_n \succ \varnothing \succ$…, the outcome is unchanged. But if these were n's true preferences, then the coalition {1,…,n-1} could make the same deviation to bring about Y, contrary to the assumed minimality of S.

Cumulative Offer Processes

- Consider a variation on the doctor-offering DA algorithm in which hospitals can go back to accept any collection of offers from the current round or any past round.
- When contracts are substitutes, the rejection function F is isotone, so if an offer is once rejected, it stays rejected.
  - That means that the Gale-Shapley doctor-offering algorithm is equivalent to one in which each hospital can go back to take any offer that she has ever received.
  - If hospitals do not regard contracts as substitutes, then
    1. Fixed points of F may not be stable.
    2. The DA algorithm may fail to terminate in a stable match.
    3. The contracts that emerge from the cumulative offer algorithm may be infeasible, assigning doctors to multiple hospitals.

Cumulative Offer Auction

- Suppose that there is just one "hospital," which is the auctioneer, so that the final match is always feasible.
- Imagine that the "doctors" are bidders who are interested in supplying packages of services to the auctioneer, which may include various services on various terms at various prices.
- The auction works like this:
  - Bidders submit rank order lists of their bids ("contracts").
  - The auctioneer submits a rank-order list of collections of bids.
  - The cumulative offer algorithm is run. At each round,
    - Each bidder that is not in the currently winning package makes the next bid on its acceptable list.
    - The auctioneer tentatively rejects all bids except its most combination, which can include at most one bid from each bidder.

Core Outcome

- **Theorem (Ausubel & Milgrom)**. The outcome of the cumulative offer auction is a core allocation.

- **Proof**. At the end of the auction,
  - each bidder has made every offer that it prefers to the final outcome, so any blocking coalition must combine the existing accumulated offers.
  - the auctioneer has accepted the combination of accumulated bids that it most prefers.

The Case of Substitutes

- **Theorem (Ausubel & Milgrom)**. If the auctioneer regards contracts as substitutes, then the outcome of the cumulative offer auction is the bidder best core allocation.

- This can be proved as a corollary of the matching with contracts result, because in the substitutes case the two algorithms coincide.

SUBSTITUTES & QUASI-LINEAR UTILITY

Aside: Submodular Minimization

- **Definition**. Given a lattice $(X,\geq)$, a function $f{:}X\rightarrow \mathbf{R}$ is ***submodular*** if for all $x,y \in X$, $f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$.

- The condition means that the incremental return to increasing one variable is declining in the other variables.
- For any smooth function $f{:}R^N\rightarrow R$, f is submodular if and only if its mixed partial derivatives are non-positive.

Indirect Utility and Substitutes

- Define the indirect utility function for a bidder by: $\pi(p) = \max_x v(x) - p \cdot x$.
- By Hotelling's lemma, the bidder's demand for good l at prices p is: $x_l(p) = -\pi_l(p)$ where the second term is the partial derivative of π with respect to the price p_l.
- Goods are therefore substitutes if and only if
  - $\pi_l(p)$ is a non-increasing function of each p_k for $k \neq l$, or equivalently
  - $\pi(\cdot)$ is submodular.

Substitutes and Prices

- If there are multiple copies of each good, market-clearing prices may fail to exist if a small increase in the price of a good can lead to either
  - Bidder reduces demand for the good by two units
  - Bidder reduces demand by one unit but substitutes two units of another product.

- **Definition (Milgrom & Strulovici)**. Goods are ***strong substitutes*** for a participant if, when each unit is treated as an individual good, these redefined goods are substitutes.
  - Kelso-Crawford ⇒ Existence of clearing prices.

Vickrey, Substitutes and the Core

- **<u>Theorem</u>.** If the goods are strong substitutes for all bidders and if the auctioneer has a fixed supply to sell, then the Vickrey auction outcome is the bidder-best core allocation.

- Role/Interpretation
  - Each bidder gets its highest payoff in the core.
  - Seller gets its lowest revenue in the core.
  - This result connects incentives in matching theory to those in ascending auctions.

Proof of Theorem

- Let v(·) be the coalitional value function constructed from the data, with zero value to coalitions that exclude the auctioneer.
- Any payoff to a bidder j higher than v(N)-v(N-j) results in an allocation that is blocked by coalition N-j, so there are no higher bidder core payoffs.
- Suppose that, as we will later show, the function v(·) is submodular. Consider any coalition S that includes the seller (player 0). Let it be S = {0,1,…,*k*}. Then, S is not a blocking coalition, because
- $$\begin{aligned}\sum_{j\in S}\pi_j &= v(N)-\sum_{j\notin S}\pi_j \\ &= v(N)-\sum_{j\notin S}\big(v(N)-v(N-j)\big) \\ &\geq v(N)-\sum_{j=k+1}^{|N|}\big(v(N-\{j+1,\ldots,|N|\})-v(N-\{j,\ldots,|N|\})\big) \\ &= v(S)\end{aligned}$$

From Goods to Bidders

- To finish, we must show that v(·) is submodular.
    - We need to use our hypothesis that indirect utility – a function of ***prices*** – is submodular to derive the conclusion that coalitional values – a function of ***sets of bidders*** – is submodular.

- **Lemma**. Suppose that for every bidder, preferences are quasi-linear and goods are strong substitutes. Then the function *v*

$$v(S) = \max_x \sum_{j \in S} v_j(x_j) \text{ subject to } \sum_{j \in S} x_j \leq \bar{x}$$

is submodular.

Aside: Submodular Minimization

- **<u>Theorem (Topkis)</u>**. If $(X \times Y, \geq)$ is a product lattice and $f: X \times Y \rightarrow \mathbf{R}$ is submodular, then $\min_y f(x,y)$ is a submodular function from $X \rightarrow \mathbf{R}$.
  - Remark. This is similar to the fact that if $f(x,y)$ is convex, then $\min_y f(x,y)$ is a convex function of x.

A Useful Submodular Function

- Let $F(p,S) = \sum_{j \in S} \pi_j(p) + p \bullet \bar{x}$.

- Let us verify that if the indirect utility functions $\pi_i(p)$ are submodular, then F is submodular:
  - S alone: $F(p,S \cup \{i\}) = F(p,S)+F(p,\{i\})$.
  - p alone: submodular $\pi_i(p)$ functions.
  - S,p: $F(p,S \cup \{i\})-F(p,S) = \pi_i(p)$ is non-increasing.

Proof of Lemma

- We calculate the coalitional value as follows:

$$
\begin{aligned}
v(S) &= \max_x \sum_{j\in S} v_j(x_j) \text{ subject to } \sum_{j\in S} x_j \le \bar{x} \\
&= \min_{p\ge 0} \max_x \sum_{j\in S} v_j(x_j) - p\bullet\left(\sum_{j\in S} x_j - \bar{x}\right) \\
&= \min_{p\ge 0} \sum_{j\in S} \max_{x_j} \left(v_j(x_j) - p\bullet x_j\right) + p\bullet\bar{x} \\
&= \min_{p\ge 0} \sum_{j\in S} \pi_j(p) + p\bullet\bar{x} \\
&= \min_{p\ge 0} F(p,S), \text{ where } F(p,S) = \sum_{j\in S} \pi_j(p) + p\bullet\bar{x}.
\end{aligned}
$$

- Because goods are strong substitutes, the market clearing price vector *p* exists, which justifies the second line above.
- Because goods are substitutes, each $\pi_j(\cdot)$ is submodular and non-increasing…
    - and these imply that *F*(*p,S*) is submodular.
- Therefore, by Topkis's theorem, *v*(·) is submodular.

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Summary and Forecast

- The "Matching with Contracts" approach exposes strong links between matching theory and both price theory and auction theory.
  - The submodular dual representation of substitutes applies to both.
- The extended substitutes condition plays a key role in all three for
  - Stable matches to exist and the GS algorithm to work
  - *Tatonnement* stability and the existence of seller-best and buyer-best equilibria (highest and lowest prices).
  - Vickrey outcomes to lie in the core and to result from cumulative offer auctions.
- Next lecture exposes more connections, including a deeper connection to Marshallian dynamics, as part of the design of the upcoming US incentive auction.

END