Common notions from hep-th

Rob Myers

Since the students at this school have very varied backgrounds, I am writing a few brief notes on a few basic (and not so basic) concepts that are part of the lingua franca on hep-th and which will crop up at various points in my lectures. Of course, any one of these could be the topic of a course in itself and I am just sketching a few basic pointers on each and providing some web resources for people who want to dig a bit deeper. However, you really should not have to have an in-depth knowledge of any of these ideas to follow the lectures. Of course, do not hesitate to let me know and ask questions (either during or after the lectures) if you feel I am leaving you behind!

1 Lorentz invariance

Implicitly, the QFT's considered in my lectures will be invariant under Lorentz transformations.

For example, using the standard relations from quantum mechanics,[1] *e.g.,* $\partial_t\phi = i[\phi, H]$ and $\partial_t\pi = i[\pi, H]$, the equation of motion for the free scalar discussed in the first lecture becomes

$$-\partial^2\phi + \sum_i \partial_i^2\phi - \mu^2\phi = 0\ .$$

We can write this equation as

$$\eta^{ab}\partial_a\partial_b\phi - \mu^2\phi = 0 \tag{1.1}$$

where we are using standard summation notation, *i.e.,* repeated indices are summed over. Further in the sums, $a, b \in \{0, 1, 2, \cdots, d-1\}$ where $x^0 = t$ and we have introduced the Minkowski 'metric'

$$\eta^{ab} = \eta_{ab} = \mathrm{diag}(-1, 1, 1, \cdots, 1)\ .$$

Hence the theory is invariant under coordinate transformations

$$\tilde{x}^a = x^b\,\Lambda_b{}^a \qquad \text{where} \quad \Lambda_a{}^c\,\Lambda_b{}^d\,\eta_{cd} = \eta_{ab}\ .$$

These transformations comprise the Lorentz group $SO(1, d-1)$, which includes the rotations amongst the $d-1$ spatial directions and boosts along each of the spatial directions. A formula we will use is that for the proper distance between to points:

$$\eta_{ab}\,\Delta x^a \Delta x^b = \sum_i (\Delta x^i)^2 - \Delta t^2\ . \tag{1.2}$$

Note that this quantity may be positive, negative or 0 depending on the relative magnitudes of the Δx^i and Δt. Of course, the result is independent of which frame it is calculated in.

One last note is that we can derive the scalar field equation above from a Lorentz invariant action

$$I = -\frac{1}{2}\int d^dx\,\left(\eta^{ab}\partial_a\phi\,\partial_b\phi + \mu^2\phi^2\right)\ . \tag{1.3}$$

That is the equation of motion is produced by extremizing $\delta I/\delta\phi = 0$.

Again Lorentz invariance is an important feature of most QFT's studied on hep-th and essential for a few points in my lectures.

[1] **Recall my lazy conventions are to set both $\hbar = 1 = c$!!**

Students who want to learn some more basics about Lorentz transformations (and special relativity) might consult:

http://en.wikipedia.org/wiki/Lorentz_transformation
http://en.wikipedia.org/wiki/Special_relativity

2 QFT's in curved spacetimes

In my lectures and in Daniel Harlow's lectures, we also encounter quantum field theories on curved spacetimes, *e.g.,* on black hole spacetime or a 'Euclidean' QFT on a d-dimensional sphere. The key here is to have a mathematical description of the new geometry on which the QFT lives. Essentially the flat space metric η_{ab} introduced above is replaced by a curved space metric $g_{ab}(x)$, where as indicated the individual components can be functions of the coordinates x^a. In analogy to eq. (1.2) which applies for finite separations, the metric g_{ab} measures the proper distance for infinitesimal displacements in the curved space geometry: $ds^2 = g_{ab}dx^a dx^b$

Within this framework, we wish to consider theories that are invariant under general coordinate transformations $y^a = y^a(x^b)$ under which the metric transforms as

$$\tilde{g}_{ab} = \frac{\partial x^c}{\partial y^a}\,\frac{\partial x^d}{\partial y^b}\,g_{cd}\ .$$

Hence we construct objects which are invariant under these transformations, *e.g.,* the equation of motion (1.1) and the action (1.3) for the free scalar field become in curved space are

$$\frac{1}{\sqrt{-g}}\,\partial_a(\sqrt{-g}g^{ab}\partial_b)\phi - \mu^2\phi = 0\ ,$$
$$I = -\frac{1}{2}\int d^dx\sqrt{-g}\left(g^{ab}\partial_a\phi\partial_b\phi + \mu^2\phi^2\right)\ .$$

Note that if we consider a Taylor expansion of the metric around a single point $x^a = x_0^b$ in the spacetime, we can always choose a coordinate transformation such that the metric reduces to the flat space metric up to second order terms in the expansion, *i.e.,*

$$g_{ab}(x) = \eta_{ab} + O((x-x_0)^2\,\partial^2 g|_{x_0})\ .$$

This means that locally the physics still looks like the physics of flat space, *i.e.,* special relativity. The fact that the second derivative terms could not all be eliminated by a suitable coordinate transformation simply reflects the fact that the spacetime is curved. These second derivatives can be collected in a covariant description of this curvature in an object know as the Riemann tensor. The details of the curvature tensor will not be important for us, although they will make a passing appearance at various points in the lecture.

Students unfamiliar with the concepts of the spacetime metric or the related concepts might consult:

http://en.wikipedia.org/wiki/Metric_%28general_relativity%29
http://en.wikipedia.org/wiki/Riemann_curvature_tensor
http://en.wikipedia.org/wiki/Quantum_field_theory_in_curved_spacetime

3 Energy-momentum tensor

The basic message here is that in relativistic theories, the energy density is a part of a symmetric two-index tensor, known as the energy-momentum tensor or the stress-energy tensor or just the stress tensor.

Given a covariant action as described above for some field theory, a simple definition of the stress tensor is

$$T_{ab} = -\frac{2}{\sqrt{-g}}\,\frac{\delta I}{\delta g^{ab}}\ .$$

Note that we are varying the action with respect to the inverse metric here. Translation invariance in flat space theories (or invariance under coordinates transformations in a curved space theories) ensures conservation the stress tensor, *i.e.,* $\nabla^a T_{ab} = 0$.

The components of the stress tensor correspond to: T_{00}, the energy density; T_{0i}, the momentum density; and T_{ij}, stresses or momentum fluxes (the diagonal component correspond to the pressure). Hence the Hamiltonian will be constructed with T_{00}, *i.e.,* $H = \int d^{d-1}x\, T_{00}$ in flat space.

In a generic non-relativistic theory, one has $|T_{00}| \ll |T_{0i}| \ll |T_{ij}|$. However, in relativisitc theories, all of the components can be comparable. For example, in a conformal field theory, we have $T^a{}_a = g^{ab}\,T_{ab} = 0$ — that is, $T_{00} = \sum_i T_{ii}$ in flat space.

Students who want to learn some more basics about the stress tensor might consult:
http://en.wikipedia.org/wiki/Energy-momentum_tensor_%28general_relativity%29

4 Path integrals

Path integrals are an alternative representation of the Hamiltonian evolution of a quantum system, which involve a (formal) integral over all configurations which might mediate the transition between two states. This approach is prevalent in the discussion of relativistic (or covariant) QFT's since it naturally provides a Lorentz invariant (or coordinate invariant) description of the evolution of the system.

The derivation of the path integral from standard Hamiltonian evolution in quantum mechanics is straightforward but lengthy enough that I will not present it here. Rather I will invite the interested students to read the webpage below.

At a couple of points in the lectures, we encounter a path integral over field configurations of some QFT on some fixed background or a closed Euclidean geometry. In analogy to statistical mechanics, we refer to these objects as the 'partition function.'

Students who want to learn some more basics about path integrals might consult:
http://en.wikipedia.org/wiki/Path_integral_formulation

5 Conformal Field Theory

In the special case that $\mu = 0$, the free scalar theory considered in the lectures and in the first section above has no intrinsic scale, *i.e.,* no dimensionful parameters. The theory also has an extra symmetry, namely, we can rescale the coordinates by a constant: $x^a \to \lambda\, x^a$. This transformation leaves the equation of motion (1.1) unchanged and also the action (1.3) if the coordinate tranformation is further accompanied by a scaling of the scalar field: $\phi \to \lambda^{-\frac{d-2}{2}}\phi$. It has been found that any relativistic theory with this extra constant scaling symmetry actually has a larger symmetry under 'conformal transformations.' The latter involve making selected coordinate transformations which lead to a 'Weyl rescaling' of the metric, *i.e.,*

$$x^a \to y^a = y^a(x^b) \qquad \text{such that} \quad g_{ab} \to \tilde{g}_{ab} = \frac{\partial x^c}{\partial y^a}\frac{\partial x^d}{\partial y^b} g_{ab} = \lambda^2(x)\, g_{ab}\ .$$

Note that scale factor is generally not a constant but may depend on the spacetime coordinates here. In a standard discussion of CFT's, we restrict ourselves to the flat space metric, *i.e.,* $g_{ab} = \eta_{ab}$. For $d > 2$ then, there are a finite set of conformal transformations including: Lorentz transformations (including the boosts and rotations), translations, constant rescaling or dilatations, and special conformal transformations (which parametrized by a vector).

However, for the most part in these lectures, we will be considering a slightly more general application where the coordinate transformations are chosen to yield $g_{ab} \to \tilde{g}_{ab} = \lambda^2(x)\, \hat{g}_{ab}$, where the new metric $\hat{g}_{ab}$ again describes some simply geometry. The transformation then involves mapping a given state of the CFT in one geometry to a new state in a new geometry. The utility of this approach will be that we may have a better understanding into the physics of the second state but by applying the inverse mapping, we will gain new insights into the physics of the original state. The transformation does not really produce a symmetry, but we might say that the theory is 'covariant' under such transformations, *i.e.,* physical objects all transform in a simple proscribed fashion.

One final comment about CFT's: Above we say that the stress tensor for a given QFT can determined by varying the action with respect to that metric. For CFT's, the action is invariant under (local) rescalings of the metric. As a consequence, it is not hard to show that the trace of the stress tensor vanishes in a CFT, *i.e.,* $(T_{\text{CFT}})^a{}_a = g^{ab}\,(T_{\text{CFT}})_{ab} = 0$. This property will hold in general for the classical field theory but as will be mentioned in the lectures, in the quantum theory, this property is not quite true in general. Rather one finds that $(T_{\text{CFT}})^a{}_a \sim$ 'curvatures' when the CFT is put in a curved background, when the dimension of the spacetime is even. This effect is known as the trace anomaly.

Students who want to learn some more basics about conformal field theories might consult:

http://en.wikipedia.org/wiki/Conformal_field_theory
http://en.wikipedia.org/wiki/Scale_invariance
http://en.wikipedia.org/wiki/Conformal_symmetry
P. H. Ginsparg, "Applied Conformal Field Theory," http://arxiv.org/abs/hep-th/9108028.