Counterterms, critical gravity and holography

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• Based on
  arXiv:1201.1288 w Kallol Sen and Nemani Suryanarayana
  work in progress with Ling-Yan Hung
• I will talk about some observations about AdS counterterms
• To obtain finite results in AdS/CFT we need to add counterterms. On the gravity side

\[ I_{tot} = I_{bulk,M}^{d+1} + I_{GH,\partial M}^d + I_{ct,\partial M}^d \]
\[ I_{\text{bulk}} = -\frac{1}{2\ell_P^{d-1}} \int_{\mathcal{M}} d^{d+1}x \sqrt{g} \left( R + \frac{d(d-1)}{L^2} \right) \]

\[ I_{\text{surf}} = -\frac{1}{\ell_P^{d-1}} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} K \]

\[ I_{\text{ct}} = \frac{1}{\ell_P^{d-1}} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} \left[ \frac{d-1}{L} + \frac{L}{2(d-2)} \mathcal{R} \right. \\
\left. + \frac{L^3}{2(d-4)(d-2)^2} \left( \mathcal{R}_{ab} \mathcal{R}^{ab} - \frac{d}{4(d-1)} \mathcal{R}^2 \right) \right] + \cdots \]

- In 4d = \frac{1}{2} (\text{Weyl}^2 - \text{Gauss-Bonnet})
- Needed to cancel divergences in d>4

Residues of the simple poles seem to encode information about trace-anomaly.

Originally, it was thought that we should only retain those terms that were needed to cancel divergences and disregard all higher order terms. Eg, \( R^2 \) terms would not play a role for d=4.
• In this talk I will connect counterterms to 2 interesting topics of current interest:
  
  1. **Critical Gravity**
  2. “**Dilaton action**” due to spontaneously broken conformal symmetry (c-theorem in arbitrary dimensions).
A very brief introduction

- Critical gravity theories are higher derivative theories whose linearized equation of motion look like

\[ \left( \square + \frac{2f_\infty}{L^2} \right)^2 h_{ab} = 0. \]

- “Dilaton action”

\[ S_{\text{anomaly}} = -a \int d^4 x \sqrt{-g} \left( \tau E_4 + 4(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial \tau)^2 \square \tau + 2(\partial \tau)^4 \right) + c \int d^4 x \sqrt{-g} \tau W_{\mu\nu\rho\sigma}^2. \]

Cf: c-theorems in arbitrary dimensions--Myers, AS; Jafferis et al
Critical gravity

- Quadratic critical gravity theories have the following form:

\[
\frac{1}{2\ell_P^{n-2}} \int d^n x \sqrt{-g} \left[ R - 2\Lambda_0 + \alpha L^2 R^2 + \beta L^2 R_{ab} R^{ab} + \gamma L^2 GB \right]
\]

- Parameters fixed by demanding linearized eoms take the form

\[
h = 0 \quad \left[ \Box + \frac{2f_\infty}{L^2} \right]^2 h_{ab} = 0.
\]

\[
\alpha = -\frac{n\beta}{4(n - 1)}
\]

eg

n=3 theory is New Massive Gravity of Bergshoeff, Hohm, Townsend
• Critical gravity is similar to the Pais-Uhlenbeck oscillator (1950)

\[ L = -\frac{1}{2} \phi \prod_{i=1}^{N} (\partial^2 - M_i^2) \phi \]

• Non-relativisitc limit

\[ L = \frac{1}{2} (\dot{q}^2 - (w_1^2 + w_2^2)q^2 + w_1 w_2 q^2) \]

• \( w_1 = w_2 \). Hamiltonian is Jordan diagonal leading to one state with zero norm. Bender-Mannheim claim that there are no ghosts if one uses rules of PT quantum mechanics.
• The reason for interest in such theories is the hope of having a unitary, renormalizable toy model of gravity.

• A generic 4-derivative theory in 4d may be renormalizable [Stelle] but will have ghosts.

• In critical gravity, the propagator changes from $1/p^2-1/(p^2-m^2)$ to $1/p^4$. So it is no longer clear if there are ghosts.

• In what I will have to say, I will be in Euclidean signature so I will not worry about such things.
• Critical gravity theories have some weird properties.
• Schwarzschild black hole still is an exact solution with finite horizon area. However the entropy of these black holes is ZERO.
• The action for empty AdS and black hole both vanish. Both vacua have equal weights.
• At the critical point, it turns out that in the holographic dual CFT, there is always one central charge that vanishes. In the existing examples in 4d it was `c'.

\[ \langle T_{ab}T_{cd} \rangle \propto c \langle 0|0 \rangle = 0 \]

• There is something called c=0-catastrophe (cf. Cardy) which necessitates existence of “logarithmic partner” to the stress tensor. Duals are log-CFTs. I will not have time to discuss this.
Comments on counterterms

• It seems somewhat unsatisfactory if for each dimension we needed to consider a specific subset of these counterterms.

• It seems more natural if there wasn’t a truncation.

• If one tried to add subleading higher-order terms to the counterterm action, one would begin to contribute to terms that vanish as the cutoff is taken to infinity. But such terms would become relevant if the cutoff is kept finite.
Counterterm action

- For $n=3$, Dileep Jatkar and I realized that the relative coefficient to get New Massive Gravity is exactly the same as what appears in the $AdS_4$ 4-derivative counterterm.

- However in $AdS_4$ the $R^2$ terms are not needed to cancel divergences. They contribute only at subleading order. In fact we realized that, that precise combination canceled off the first subleading cutoff dependence.
We asked the following question:

Is it possible to cancel all the cutoff dependence in the Euclidean AdS action?

We found the following simple answer (later we will allow for a one parameter gen.):

\[ I_{ct} = -\frac{2L^2}{\ell_P^2} \sqrt{-\det(\mathcal{R}_{ab} - \frac{1}{2} \mathcal{R} \gamma_{ab} - \frac{1}{L^2} \gamma_{ab})} \]

Canceled cutoff dependence for euclidean action in AdS for all boundary topologies.

Reminiscent of a “DBI” type action but involving gravity

DBI gravity; in arXiv, first considered by Deser and Gibbons in flat space. But Eddington and Schrodinger have mused about similar replacements of GR.
• Can rewrite this theory in terms of a Chern-Simons gauge field

\[ A^{a \pm} = \omega^{a \pm} \pm \frac{1}{\ell} e^{a} \]

\[ I_{ct} \propto \sqrt{\text{det} \star F_{a \mu}} \]

• So simple!

• It is an easy calculation to take JUST the counterterm action put it on an AdS with \( S^2 \) boundary. Action vanishes: central charge (of dual 1+1CFT) vanishes.
• This leads to the suspicion that these counterterm actions treated as independent actions are just critical gravity theories.

• Prompts need to do a careful analysis in higher dimensions to see how general this is.
• With Kallol S. and Nemani S., we generalized this result to arbitrary odd dimensional CFTs. First in AdS$_4$, there was a one parameter extension:

$$I_{ct}^{(3)} = \frac{3}{L \ell_P^2} \int d^3 x \sqrt{\gamma} \left( 1 + \frac{1}{2} L^2 \mathcal{R} - \frac{1}{2} L^4 (\mathcal{R}_2 - \frac{1}{2} \mathcal{R}^2) + \lambda L^6 \mathcal{R}^3 + \left( \frac{1}{24} - 5\lambda \right) L^6 \mathcal{R} \mathcal{R}_2 + (6\lambda - \frac{1}{12}) L^6 \mathcal{R}_3 \right)^{1/2}$$

• The 1+1 dual still had zero central charge.
• Similar actions for AdS$_6$ and AdS$_8$ which canceled cutoff dependence for the Euclidean action with $S^d$ and $S^d \times S^1$ boundaries (Weyl flat).

• Linearized equation of motion

\[
\frac{(d - 2)^2(d - 1)}{2d} c_1 \Box h = 0
\]

\[
c_1 \left( -2 \frac{(d - 1)^2}{d} L^2 \nabla_a \nabla_b h + \frac{2}{d} h g_{ab} - L^4 \frac{1}{2} (\Box + \frac{2}{L^2}) \Box \left( \Box + \frac{2}{L^2} \right) h_{ab} \right) = 0. 
\]
• After field redefinition

\[ \hat{h}_{ab} = h_{ab} - \frac{h}{d} \tilde{g}_{ab} + \frac{1}{d} L^2 \nabla_a \nabla_b h \]

\[ \frac{c_1}{2} \left( \square + \frac{2}{L^2} \right) \left( \square + \frac{2}{L^2} \right) \hat{h}_{ab} = 0 \]

• Takes the form in critical gravity

• In other words the counterterm actions are actions of critical gravity. They have an infinite set of higher derivative corrections. The Euler anomaly of the dual CFT to the counterterm theory is zero. Very likely the CFT is a log-CFT.
• We found that

\[ I_{ct} = -(d - 1) \frac{L^{d-1}}{\ell_P^{d-1}} \left( -\frac{1}{\prod_i \lambda_i} \frac{1}{d!} \epsilon^{a_1 \ldots a_d} \epsilon^{b_1 \ldots b_d} \prod_{i=1}^d G_{a_i b_i}^{(i)} \right)^{1/2} \]

where

\[ G^{(i)} = \mathcal{R}_{ab} - \frac{1}{d-1} \mathcal{R} \gamma^{ab} + \frac{\lambda_i}{L^2} \gamma^{ab}. \]

\[ f(\lambda = \frac{L^2}{N^2})|_{\partial M = S^d} = (I_{\text{bulk}} + I_{GH})^2 = 0. \]

cancels cutoff dependence in any odd dimensional CFT. Same formula cancels log-independent cutoff dependence in even dimensions. \( \lambda_i \)'s are real only upto \( d+1=7 \) (Recall AdS\(_{d+1}\) for \( d>6 \) do not arise in string theory)!
Connection to c theorem in odd dimensions: A quick review

Consider AdS$_4$ bulk

\[
\frac{I_{\text{bulk}}}{V_{S^3}} = \frac{3}{\ell_P^2} V_{S^3} \int_0^\Lambda dr \frac{r^3}{\sqrt{1 + \frac{r^2}{L^2}}}
\]

\[
\frac{I_{\text{bulk}}}{V_{S^3}} = \frac{L^2}{\ell_P^2} \left( \frac{\Lambda^2}{L^2} - 2 \right)(1 + \frac{\Lambda^2}{L^2})^{1/2} + 2 \frac{L^2}{\ell_P^2}
\]

\[
\frac{I_{\text{GH}}}{V_{S^3}} = -3 \frac{\Lambda^2}{\ell_P^2} (1 + \frac{\Lambda^2}{L^2})^{1/2}
\]

\[
\frac{I_{\text{bulk}} + I_{\text{GH}}}{V_{S^3}} = -2 \frac{L^2}{\ell_P^2} (1 + \frac{\Lambda^2}{L^2})^{3/2} + 2 \frac{L^2}{\ell_P^2}
\]

Free energy.
Comes from lower limit in bulk integral

Counterterm cancels this exactly!
Now observe that the counterterm seems to know about the **finite free energy piece**. Simply consider a small cutoff expansion!

We will see in a bit that in even dim-CFTs it is possible to extract the anomaly from the counterterms which cancel power law divergences. Here we see that it is possible to extract the free energy from the counterterms!

We had to work very hard to get an expression that was valid for all cutoff. In other words:

\[
\frac{I_{\text{bulk}} + I_{GH}}{V_{S^3}} = -2 \frac{L^2}{\ell_P^2} (1 + \frac{\Lambda^2}{L^2})^{\frac{3}{2}} + 2 \frac{L^2}{\ell_P^2}
\]

To get the finite piece for the 3d-CFT from counterterms we needed to resum an infinite number of terms
Anomalies from counterterms

[work in progress with Ling-Yan Hung]

• Let us go back to the derivative expansion of the counterterm action+Gibbons-Hawking for even $d$

\[
I_{ct} = \frac{1}{\ell_P^{d-1}} \int_{\partial \mathcal{M}} d^d x \sqrt{\gamma} \left[ \frac{d - 1}{L} + \frac{L}{2(d - 2)\mathcal{R}} \right. \\
\left. + \frac{L^3}{2(d - 4)(d - 2)^2} \left( \mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{d}{4(d - 1)\mathcal{R}^2} \right) + \cdots \right]
\]

• Write bulk as

\[
\frac{1}{z^2} (dz^2 + dx \cdot dx)
\]
• Set (radion field)

\[ z = U(x) = e^{\sigma(x)} \]

• Motivation for doing this? Imagine a connection of counterterms with brane action. The radion field captures centre of mass motion of brane

(singletons??)

• Consider a derivative expansion of counterterm action in \( \sigma(x) \).
• In even dimensions results are:

\[ d=2 \quad \frac{L}{\ell_P} \left[ -\frac{1}{2}(\partial \sigma)^2 \right] \]

\[ d=4 \quad \frac{L^3}{\ell_P^3} \left[ -\frac{1}{8}(\partial \sigma)^4 + \frac{1}{4}(\partial \sigma)^2 \partial^2 \sigma \right] \]

\[ d=6 \quad \frac{L^5}{\ell_P^5} \left[ -\frac{1}{8}(\partial \sigma)^6 + \frac{1}{16}(\partial_a \partial_b \sigma)^2 (\partial \sigma)^2 \right] \]

• Using these expressions, we can extract the anomalies by comparing with the relevant piece of the “dilaton action”. They match exactly with what we expect!
• In other words we seem to be able to extract trace anomalies from the “power-law” counterterms.

• One can hope that some clever resummation exists as indicated by our construction in holography that captures the free energy from counterterms in odd dimensions. Note that from holography the “resummed” answer is known (DBI-generalized form). Question is if we can argue that this is precisely its form independent of holography.
Discussion

• Counterterm actions are intriguing. They are related to critical gravity.

• Finite cutoff counterterms were shown. Did not have time to show that these would lead to consistency with the first law of black hole thermodynamics.

• Seem to be able to extract anomalies from “power law divergences”.

• What about odd-d c-theorem? Any hope? The free energy is captured by the counterterms. But needed to do a resummation. May be something similar needs to be done with the dilaton action.